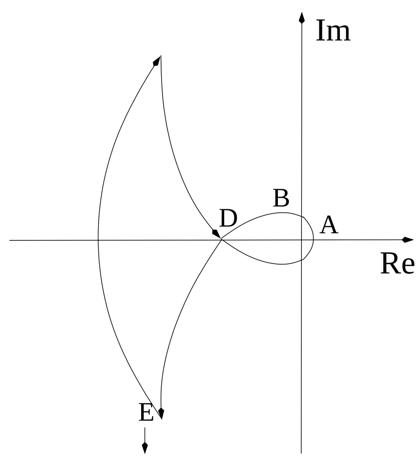
Department of Chemical Engineering
IIT Bombay
CL692, Digital Control
Assignment 7
Handed out on: 19 Sep 2006
To be completed by: 25 Sep 2006

1. We have $P = 1$. Note that we have to redo only the part corresponding to the small semicircle, *i.e.,* along the indentation next to the pole at 1. When G is evaluated along this small semicircle, we obtain

$$G(1 + \varepsilon e^{j\phi}) = \infty e^{-j\theta}$$

We have to be careful about the direction of encirclement, though. Although ϕ starts at $+90^o$ only, it now goes through 180^o, before coming to -90^o. As a result, the C_4 curve starts at -90^o, goes through -180^o, before reaching $+90^o$. The Nyquist plot is shown below:



Because we have $P = 1$, we should have $N = 1$ for stability. This will happen when $(-1/K, 0)$ is placed to the left of the point D on the real axis. Thus, once again, we get the identical stability condition.

2. Solution:
The 2×2 transfer function is given as,

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+GG_c} & -\dfrac{G}{1+GG_c} \\ \dfrac{G_c}{1+G_cG} & \dfrac{1}{1+G_cG} \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

The following are equivalent:

(a) Every term in this 2×2 matrix has to be stable for internal stability.

(b) If there is no unstable pole-zero cancellation, and if it is externally (*i.e.,* between r and y) stable, the system is internally stable.

(c) If there is no unstable pole-zero cancellation, and if S is stable, the system is internally stable.

(d) If the unstable poles of G are cancelled by zeros of G_c, S will be nonzero at those locations. If they are not cancelled, S will become zero at those places, because, G will be infinite. Thus, we arrive at the condition of (d).

(e) If the nonminimum phase zeros of G are cancelled by poles of G_c at the same locations, S will be different from 1 at those locations. If instead, there is no such unstable pole-zero cancellation, S will be 1 at these z_j, because, G will be zero.

(f) We summarize our findings: The system is internally stable, if all of the following are fulfilled.

   i. S is stable

   ii. $S(p_i) = 0$, $n \geq i \geq 1$, where, p_i are the unstable poles of G.

   iii. $S(z_j) = 0$, $m \geq j \geq 1$, where, z_j are the nonminimum zeros of G.

3. Suppose that the desired error to a step input is of the form given in Eq. 7.91. Taking Z-transform, we get,

$$E(z) = \frac{z(z - \rho\cos\omega)}{z^2 - 2z\rho\cos\omega + \rho^2}$$

As the corresponding input $R(z)$ is a unit step, the transfer function between $R(z)$ and $E(z)$ is,

$$T_E(z) = \frac{E(z)}{R(z)} = \frac{z(z - \rho\cos\omega)}{z^2 - 2z\rho\cos\omega + \rho^2}\frac{z-1}{z}$$

Simplifying, we obtain

$$T_E(z) = \frac{(z-1)(z - \rho\cos\omega)}{z^2 - 2z\rho\cos\omega + \rho^2}$$

The expression for error $e(n)$ is given by the following expression:

$$E(z) = T_E(z)R(z)$$

where, $R(z)$ is given by

$$R(z) = \frac{T_s z}{(z-1)^2}$$

Substituting, the expression for error becomes

$$E(z) = \frac{T_s z(z - \rho\cos\omega)}{(z^2 - 2z\rho\cos\omega + \rho^2)(z-1)}$$

By splitting into partial fraction expansion,

$$\frac{z-\rho\cos\omega}{(z^2-2z\rho\cos\omega+\rho^2)(z-1)}=\frac{A}{z-e^{j\omega}}+\frac{A^*}{z-e^{-j\omega}}+\frac{B}{z-1}$$

Multiplying by $z-1$ and letting $z=1$, we obtain,

$$B=\frac{1-\rho\cos\omega}{1-2\rho\cos\omega+\rho^2}$$

Applying the final value theorem, we obtain,

$$e(n)=\lim_{z\to 1}(z-1)E(z)=T_sB|_{z=1}$$

from which, the desired result follows.