Multiplying both sides of Eq. 6.81 by $v(n-2)$, taking expectation and substituting for $\gamma_{v\xi}(0)$,

$$\gamma_{vv}(2) = c_2\gamma_{v\xi}(0) = c_2\sigma_\xi^2$$

Now, we are in a position to calculate the scaled ACF:

$$\rho_{vv}(2) = \frac{\gamma_{vv}(2)}{\gamma_{vv}(0)} = \frac{c_2}{1+c_1^2+c_2^2}$$
$$\rho_{vv}(1) = \frac{\gamma_{vv}(1)}{\gamma_{vv}(0)} = \frac{c_1(1+c_2)}{1+c_1^2+c_2^2}$$

Given experimental data, we can calculate the scaled ACF, using a procedure, such as the one given in Example 6.67. We can then solve the above two equations simultaneously for c_1 and c_2.

∎

6.4.4 Condition for Unique Estimation

We would like to address the question of whether we can always determine the model parameters of an MA process from the experimental data, as explained in the previous section. The MA process, given by Eq. 6.68 implies,

$$v(n) = c(n) * \xi(n)$$

where, $c(n)$ is the inverse Z-transform of $C(z)$ and $\xi(n)$ is the white noise of variance 1. We also obtain,

$$v(-n) = c(-n) * \xi(-n)$$

Convolving the expressions for $v(n)$ and $v(-n)$ and using the commutativity property of convolution, we obtain,

$$v(n) * v(-n) = c(n) * c(-n) * \xi(n) * \xi(-n)$$

Using the definition of auto covariance, as given by Eq. 6.60, we obtain,

$$\gamma_{vv}(n) = c(n) * c(-n) * \gamma_{\xi\xi}(n)$$

Taking Z-transform of both sides, we obtain,

$$\Phi_{yy}(z) = C(z)C(z^{-1}) \tag{6.82}$$

where, we have used the fact that $\gamma_{\xi\xi}(n) = \delta(n)$ from Eq. 6.36 on page 154, for white noise of variance 1. We have also made use of the result of Sec. 4.2.8 to arrive at $C(z^{-1})$. The power of z^{-1} as an argument of C indicates that we have to replace the occurrences of z in $C(z)$ with z^{-1}.

Because the zeros of $C(z)$ are reciprocals of the corresponding zeros of $C(z^{-1})$, we should expect a loss in uniqueness. We illustrate this idea with a simple example.

Example 6.13 Study the autocorrelation function of two noise processes $v_1(n)$ and $v_2(n)$, modelled as,

$$\begin{aligned} v_1(n) &= \xi(n) + c_1\xi(n-1) = (1 + c_1 z^{-1})\xi(n) \\ v_2(n) &= \xi(n) + c_1^{-1}\xi(n-1) = (1 + c_1^{-1} z^{-1})\xi(n) \end{aligned}$$

where, we have used the mixed notation of Sec. 6.4.1. Using Eq. 6.82, we obtain the spectrum of v_2 as

$$\Phi_{v_2 v_2} = (1 + c_1^{-1} z^{-1})(1 + c_1^{-1} z)$$

In a similar way, the spectrum of v_1 is

$$\Phi_{v_1 v_1} = (1 + c_1 z^{-1})(1 + c_1 z)$$

Pulling out $c_1 z^{-1}$ and $c_1 z$, respectively, from the first and second terms of the right hand side, we obtain

$$\Phi_{v_1 v_1} = c_1 z^{-1}(c_1^{-1} z + 1) c_1 z (c_1^{-1} z^{-1} + 1)$$

Comparing this with the expression for $\Phi_{v_2 v_2}$, we obtain,

$$\Phi_{v_1 v_1} = c_1^2 \Phi_{v_2 v_2}$$

It is clear that the autocorrelation of v_1 and v_2 are identical, *i.e.,*

$$\rho_{v_1 v_1}(i) = \rho_{v_2 v_2}(i), \quad \forall i$$

because, scaling results in removal of constant factors - see the definition of ACF in Eq. 6.28 on page 152. As a result, given the autocorrelation function, it is not possible to say whether the underlying noise process is v_1 or v_2.
∎

In the above example, if c_1 lies outside it, c_1^{-1} will lie inside the unit circle. Because we can't say which one has given rise to ACF, by convention, we choose the zeros that are inside the unit circle. Although this discussion used a first degree polynomial C, it holds good even if the degree is higher. We illustrate these ideas with a Matlab based example.

Example 6.14 The MA(2) process described by

$$\begin{aligned} v_1(n) &= \xi(n) - 3\xi(n-1) + 1.25\xi(n-2) = (1 - 3z^{-1} + 1.25z^{-2})\xi(n) \\ &= (1 - 0.5z^{-1})(1 - 2.5z^{-1})\xi(n) \end{aligned}$$

is used to generate data as in M 6.7. The same code determines the model parameters. We obtain the following model:

```
Discrete-time IDPOLY model: v(t) = C(q)e(t)
C(q) = 1 - 0.8923 (+-0.009942) q^-1 + 0.1926 (+-0.009935) q^-2
```

Note that the identified model parameters are different from the ones used to generate the data. Observe also that one of the zeros lies outside the unit circle. We repeat this exercise with the process described by,

$$\begin{aligned} v_2(n) &= \xi(n) - 0.9\xi(n-1) + 0.2\xi(n-2) = (1 - 0.9z^{-1} + 0.2z^{-2})\xi(n) \\ &= (1 - 0.5z^{-1})(1 - 0.4z^{-1})\xi(n) \end{aligned}$$

Note this process is identical to v_1, but for the zero outside the unit circle (2.5) being replaced by its reciprocal (0.4). M 6.7 generates data for this model as well, and estimates the parameters. We obtain the following result:

```
Discrete-time IDPOLY model: v(t) = C(q)e(t)                  \
C(q) = 1 - 0.8912 (+-0.009939) q^-1 + 0.1927 (+-0.009935) q^-2
```

Observe that Matlab estimates the parameters correctly this time. ∎

We emphasize that the zeros of the identified polynomial will be inside the unit circle.

From this section, we conclude that ACF can be used as an effective tool to determine the order of MA processes. We will now devote our attention to AR processes.

6.4.5 Determination of Order of AR Processes

In this section, we will present a method to determine the order of AR processes. Let us first explore whether it is possible to do this through ACF. We will begin with a simple example.

Example 6.15 Calculate the ACF of AR(1) process:

$$v(n) + a_1 v(n-1) = \xi(n)$$

Multiplying both sides of this equation successively by $v(n-1)$, $v(n-2)$, ..., $v(n-l)$ and taking expectation, we obtain,

$$\begin{aligned} \gamma_{vv}(1) + a_1\gamma_{vv}(0) &= 0 \\ \gamma_{vv}(2) + a_1\gamma_{vv}(1) &= 0 \\ &\vdots \\ \gamma_{vv}(l) + a_1\gamma_{vv}(l-1) &= 0 \end{aligned}$$

where, the right hand side of every equation is zero, because of the causality condition, given by Eq. 6.33 on page 154. Starting from the last equation and recursively working upwards, we obtain,

$$\begin{aligned} \gamma_{vv}(l) &= -a_1\gamma_{vv}(l-1) = -a_1(-a_1\gamma_{vv}(l-2)) = a_1^2\gamma_{vv}(l-2) \\ &= \cdots = (-1)^l a^l \gamma_{vv}(0) \end{aligned}$$