Department of Chemical Engineering
IIT Bombay
CL692, Digital Control
Solution to Assignment 4
Handed out on: 21 Aug 2006
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1. (a)

$$\begin{aligned} u(k) &= M\cos\omega k \\ U(z) &= \frac{M}{z}\left(\frac{z}{z-e^{-j\omega}}+\frac{z}{z-e^{-j\omega}}\right) \\ \therefore\ Y(z) &= \frac{M}{z}\left(\frac{z}{z-e^{j\omega}}+\frac{z}{z-e^{-j\omega}}\right)G(z) \end{aligned}$$

where $G(z)$ is the transfer function of the system.

(b) Let

$$G(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

with $|p_i| < 1$ as G(z) is stable.Partial fraction expansion of Y(z) is next found.

$$Y(z) = \frac{\alpha z}{z-e^{j\omega}}+\frac{\beta z}{z-e^{j\omega}}+\sum_{i=1}^{n}\frac{D_i z}{z-p_i}$$

Each term in the summation will have a term of the form $D_i P_i^k$ with $|p_i| < 1$. Thus as k $\rightarrow\ \infty$, the terms in the sum vanish. The steady state reponse is from the first two terms only.

$$\begin{aligned} \alpha &= \lim_{z to e^{j\omega}} Y(z)\frac{z-e^{j\omega}}{z} \\ &= \lim_{z to e^{j\omega}}\left[\frac{M}{z}\frac{z-e^{j\omega}}{z-e^{j\omega}}G(z)\right] \\ &= \frac{M}{z}G\left(e^{j\omega}\right) \end{aligned}$$

β will be a complex conjugate of α and will be given by

$$\beta = \frac{M}{z}G\left(e^{j\omega}\right)$$

(c) Thus, the steady state response is given by

$$Y(z) = \frac{M}{z}\left[\frac{G\left(e^{j\omega}\right)z}{z-e^{j\omega}}+\frac{G\left(e^{-j\omega}\right)z}{z-e^{-j\omega}}\right]$$

As $G\left(e^{j\omega}\right)$ is a complex number, it can be written as $\left|G\left(e^{j\omega}\right)\right| e^{j\varphi}$, where φ is the angle of G.
The above equation becomes,

$$Y(z) = \frac{M}{z}\left|G\left(e^{j\omega}\right)\right|\left[\frac{e^{j\varphi z}}{ze^{j\omega}} + \frac{e^{-j\varphi z}}{z - e^{-j\omega}}\right]$$

(d) On inverting we get

$$\begin{aligned} y(k) &= \frac{M}{z}\left|G\left(e^{j\omega}\right)\right|\left(e^{j\varphi}e^{j\omega k} + e^{-j\varphi}e^{-j\omega k}\right) \\ &= \frac{M}{z}\left|G\left(e^{j\omega}\right)\right|\left(e^{j(\varphi+\omega k)} + e^{-j(\varphi+\omega k)}\right) \\ &= \frac{M}{s.s.}\left|G\left(e^{j\omega}\right)\right|\cos\left(\omega k + \varphi\right) \end{aligned}$$

Thus the output also is a sinusoid.

(e) we removed the effect of the terms in the summation while doing partial fraction expansion of Y(z). If G(z) were not stable, the s.s. response may not exist, let alone it being a sinusoid of the same frequency as the input.

2. (a)

$$\begin{aligned} y(n) &= g(n) * u(n) = \sum_{i=-\infty}^{\infty} g(i)u(n-i) \\ &= \delta(0)u(n) - \sqrt{2}\delta(0)u(n-1) + \delta(0)u(n-2) \\ &= u(n) - \sqrt{2}u(n-1) + u(n-2) \\ y_1(n) &= \cos\frac{\pi}{4}1(n) - \sqrt{2}\cos\frac{\pi}{4}(n-1)1(n-1) \\ &+ \cos\frac{\pi}{4}(n-2)1(n-2) \end{aligned}$$

For $n \geq 2$,

$$\begin{aligned} y_1(n) &= \cos\frac{\pi}{4}n + \cos\frac{\pi}{4}(n-2) - \sqrt{2}\cos\frac{\pi}{4}(n-1) \\ &= 2\cos\frac{\frac{\pi}{4}n + \frac{\pi}{4}(n-2)}{2}\cos\frac{\frac{\pi}{4}n - \frac{\pi}{4}(n-2)}{2} - \sqrt{2}\cos\frac{\pi}{4}(n-1) \\ &= 2\cos\frac{\frac{\pi}{2}n - \frac{\pi}{2}}{2}\cos\frac{\pi}{4} - \sqrt{2}\cos\frac{\pi}{4}(n-1) \\ &= 0 \\ y_1(0) &= 1 \\ y_1(1) &= \cos\frac{\pi}{4} - \sqrt{2} = \frac{\sqrt{2}}{2} - \sqrt{2} = \sqrt{2}\left(\frac{1}{2} - 1\right) = -\frac{1}{\sqrt{2}} \\ y_2(n) &= \cos\frac{\pi}{2}n - \sqrt{2}\cos\frac{\pi}{2}(n-1)1(n-1) + \cos\frac{\pi}{2}(n-2)1(n-2) \end{aligned}$$

For $n \geq 2$,

$$\begin{aligned} y_2(n) &= \cos\frac{\pi}{2}n + \cos\frac{\pi}{2}(n-2) - \sqrt{2}\cos\frac{\pi}{2}(n-1) \\ &= 2\cos\frac{\pi}{2}(n-1)\cos\pi 2 - \sqrt{2}\cos\frac{\pi}{2}(n-1) \\ &= -\sqrt{2}\cos\frac{\pi}{2}(n-1) = -\sqrt{2}\sin\frac{n\pi}{2} \\ y_2(0) &= 1,\ y_2(1) = -\sqrt{2},\ y(0) = 2,\ y(1) = -\sqrt{2} - \frac{1}{\sqrt{2}} \\ y(n) &= -\sqrt{2}\cos\frac{\pi}{2}(n-1),\ n\ \geq\ 2 \end{aligned}$$

(b)

$$G(z) = 1 - \sqrt{2}z^{-1} + z^{-2} = \frac{z^2 - \sqrt{2}z + 1}{z^2}$$

zeros are at $z^2 - \sqrt{2}z + 1 = 0$

$$\begin{aligned} z &= \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm \sqrt{2}j}{2} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}j \\ &= e^{\pm j\frac{\pi}{4}} \end{aligned}$$

Therefore, has a zero at $\omega = \frac{\pi}{4}$. This is the reason why $\cos\frac{\pi}{4}n$ gets filtered.

(c)

$$\begin{aligned} G\left(e^{j\omega}\right) &= 1 - \sqrt{2}e^{-j\omega} + e^{-j2\omega} \\ &= 1 - \sqrt{2}[\cos\omega - j\sin\omega] + [\cos 2\omega - j\sin 2\omega] \\ &= [1 - \sqrt{2}\cos\omega + \cos 2\omega] + j[\sqrt{2}\sin\omega - \sin 2\omega] \\ \left|G\left(e^{j\omega}\right)\right|^2 &= [1 - \sqrt{2}\cos\omega + \cos 2\omega]^2 + [\sqrt{2}\sin\omega - \sin 2\omega]^2 \end{aligned}$$

ω	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
G^2	$\left(1-\sqrt{2}+1\right)^2 + 0$ $= 0.343$	$(1-1)^2 + (1-1)^2$ $= 0$	$(1-1)^2 + \left(\sqrt{2}\right)^2$ $= 2$
G	$2-\sqrt{2}$	0	$\sqrt{2}$
$\angle G$	0	0	$\frac{\pi}{2}$

We expect $\cos\frac{\pi}{4}n$ to be rejected completely. For $\cos\frac{\pi}{2}n$, should expect the output to be

$$\begin{aligned} \sqrt{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{2}\right) &= -\sqrt{2}\sin\frac{\pi}{2}n \\ \cos\frac{\pi}{2}(n-1) &= \cos\frac{\pi}{2}n\cos\frac{\pi}{2} + \sin\frac{\pi}{2}n\sin\frac{\pi}{2} = \sin\frac{\pi}{2}n \end{aligned}$$

we previously calculated

$$y(n) = -\sqrt{2}\cos\frac{\pi}{2}(n-1) = -\sqrt{2}\sin\frac{\pi}{2}n$$

Thus this agrees with the result from Bode plot. Let us now draw the Bode plot.

$$\begin{aligned}
\left|G\left(e^{j\omega}\right)\right|^2 &= (1-\sqrt{2}\cos\omega+\cos 2\omega)^2+(\sqrt{2}sin\omega-\sin 2\omega)^2\\
&= 1+2\cos^2\omega+\cos^2 2\omega-2\sqrt{2}\cos\omega-2\sqrt{2}\cos\omega\cos 2\omega\\
&+2\cos 2\omega+2\sin^2\omega+\sin^2\omega-2\sqrt{2}\sin\omega\sin 2\omega\\
&= 1+2+1-2\sqrt{2}\cos\omega-2\sqrt{2}(\cos 2\omega-\cos\omega)+2\cos 2\omega\\
&= 4-4\sqrt{2}\cos\omega+2\cos 2\omega\\
&= 4-4\sqrt{2}\cos\omega+4\cos^2\omega-2\\
&= 2+4\cos^2\omega-4\sqrt{2}\cos\omega=(\sqrt{2}-2\cos\omega)^2\\
\left|G\left(e^{j\omega}\right)\right| &= \sqrt{(\sqrt{2}-2\cos\omega)^2}=\sqrt{2}-2\cos\omega
\end{aligned}$$

We will next calculate the angle.

$$\begin{aligned}
\angle G\left(e^{j\omega}\right) &= \tan^{-1}\left[\frac{\sqrt{2}\sin\omega-\sin 2\omega}{1-\sqrt{2}\cos\omega+\cos 2\omega}\right]\\
&= \tan^{-1}\left[\frac{(\sqrt{2}-2\cos\omega)\sin\omega}{1-\sqrt{2}\cos\omega+2\cos^2\omega-1}\right]\\
&= \tan^{-1}\left[\frac{(\sqrt{2}-\cos\omega)\sin\omega}{(-\sqrt{2}+2\cos\omega)\cos\omega}\right]\\
&= \tan^{-1}\left[-\tan\omega\right]=-\omega
\end{aligned}$$

Let us calculate some values.

$$\begin{aligned}
\text{at } \frac{\pi}{4},\ \left|G\left(e^{j\omega}\right)\right| &= \sqrt{2}-2-\frac{1}{\sqrt{2}}=0\\
\text{at } \frac{\pi}{2},\ \left|G\left(e^{j\omega}\right)\right| &= \sqrt{2}-2\cos\frac{\pi}{2}=\sqrt{2}\\
\angle G\left(e^{j\omega}\right) &= -\omega|_{\frac{\pi}{2}}=-\frac{\pi}{2}\\
phase &= -\frac{\pi}{2}\\
y(n) &= \left[\cos\frac{\pi}{4}n\right]0+\sqrt{2}\cos\left[\frac{\pi}{2}n-\left(-\frac{\pi}{2}\right)\right]\\
&= \sqrt{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{2}n\right)=-\sqrt{2}\sin\frac{\pi}{2}
\end{aligned}$$

This is in agreement with what we expect.