Department of Chemical Engineering
IIT Bombay
CL692, Digital Control
Assignment 3
Handed out on: 14 Aug 2006
To be completed by: 21 Aug 2005

1. (a)

$$u(n) = 1(n) * 1(n) = \sum_{k=-\infty}^{\infty} 1(k)1(n-k) = \sum_{k=0}^{\infty} 1(n-k)$$
$$= \left(\sum_{k=0}^{n} 1(n) \right) 1(n) = (n+1)1(n)$$
$$U(z) = Z\{u(n)\} = Z\{1(n)\}Z\{1(n)\} = \frac{z^2}{(z-1)^2}$$

(b)

$$u(n) = n.1(n) + 1(n)$$

Using the Z-transforms of $n1(n)$ and $1(n)$,

$$U(z) = \frac{z}{(z-1)^2} + \frac{z}{(z-1)} = \frac{z^2}{(z-1)^2}$$

which agrees with the result obtained through the other method.

2. (a)

$$y(n) - y(n-1) = u(n)$$

Taking Z-transforms,

$$Y(z) - z^{-1}Y(z) = U(z)$$

therefore

$$Y(z) = \frac{1}{1-z^{-1}} \; U(z) = \frac{z}{z-1} U(z)$$

(b) Since

$$y(n) = u(n) * g(n) = \sum_{k=-\infty}^{\infty} u(k)g(n-k)$$

it is clear that

$$\begin{aligned} g(n-k) &= 1 \quad \forall k \le n \\ &= 0 \quad \forall k > n \end{aligned}$$

It follows that

$$\begin{aligned} g(n) &= 1(n) \\ y(n) &= u(n) * 1(n) \end{aligned}$$

or

$$Y(z) = \frac{z}{z-1} U(z)$$

which agrees with the result obtained through the earlier method.

3.

$$\begin{aligned}
G(z) &= \frac{1}{(1-az^{-1})(1-bz^{-1})} \\
&= \frac{a}{a-b}\frac{1}{1-az^{-1}} + \frac{b}{b-a}\frac{1}{1-bz^{-1}} \\
g(n) &= \frac{a}{a-b}a^n 1(n) + \frac{b}{b-a}b^n 1(n) \\
g(0) &= \frac{a}{a-b} - \frac{b}{b-a} = 1 \\
g(1) &= \frac{a^2}{a-b} + \frac{b^2}{-(a-b)} = \frac{a^2-b^2}{a-b} = a+b
\end{aligned}$$

<u>Alitar</u>

$$\begin{aligned}
g_1(n) * g_2(n) &= a^n 1(n) * b^n 1(n) \\
g(n) &= \sum_{k=0}^{\infty} a_1^k 1(k) b^{n-k} 1(n-k) \\
&= \left[b^n \sum_{k=0}^{n} \left(\frac{a^k}{b^k} \right) \right] 1(n) \\
&= b^n \frac{1-(a/b)^{n+1}}{1-a/b} 1(n) = \frac{b^{n+1}-a^{n+1}}{b-a}
\end{aligned}$$

From the above expression, $g(0)$ and $g(1)$ values can be verified.
<u>Alitar</u>

$$\begin{aligned}
g_1(n) &= a^n 1(n) \\
g_2(n) &= b^n 1(n) \\
g_1(n) * g_2(n) &= \{1, a, a^2, ...\} * \{1, b, b^2, ...\} \\
&= \{1, (a+b), (a^2+ab+b^2), ...\} \\
g(0) &= 1, g(1) = (a+b)
\end{aligned}$$

<u>Alitar</u>

$$\begin{aligned}
G(z) &= \frac{1}{1-(a+b)z^{-1}+ab^{-2}} \\
&= \frac{z^2}{z^2-(a+b)z+ab}
\end{aligned}$$

which is obtained by long division, carried out below.

$$g(0) = 1, g(1) = (a+b)$$

<u>long division</u>

$$\begin{array}{r|l}
 & 1+(a+b)z^{-1} \\
\hline
z^2-(a+b)z+ab & z^2 \\
 & z^2 \quad -(a+b)z+ab \\
\hline
 & \quad (a+b)z-ab \\
 & \quad (a+b)z-(a+b)+... \\
\hline
\end{array}$$

Alitar

$$
\begin{aligned}
g(0) &= \lim_{z\to\infty} \frac{z^2}{(z-a)(z-b)} = 1 \\
g(1) &= \lim_{z\to\infty} \frac{z^2(a+b)+abz}{(z-a)(z-b)} = \lim_{z\to\infty} \frac{(a+b)+ab/z}{(1-a/z)(1-b/z)} \\
&= a+b
\end{aligned}
$$

4. $|z|>2$:

$$
G(z) = \frac{z^2+2z}{(z+1)^2(z-2)} \quad |z|>2
$$

Using the result from Example 4.27, we obtain

$$
\begin{aligned}
G(z) &= -\frac{4}{9}\frac{z}{z+1} - \frac{1}{3}\frac{z}{(z+1)^2} + \frac{4}{9}\frac{z}{z-2} \\
g(n) &= -\frac{4}{9}(-1)^n 1(n) - \frac{1}{3}n(-1)^{n-1} 1(n) + \frac{4}{9}2^n 1(n)
\end{aligned}
$$

$|z|<1$:

$$
G(z) = -\frac{4}{9}\frac{z}{z+1} - \frac{1}{3}\frac{z}{(z+1)^2} + \frac{4}{9}\frac{z}{z-2} \quad |z|<2
$$

From Ex.4.2,

$$
\frac{z}{z-a} \leftrightarrow -a^n 1(-n-1)
$$

Therefore,

$$
-\frac{4}{9}\frac{z}{z+1} \leftrightarrow \frac{4}{9}(-1)^n 1(-n-1)
$$

From section(4.4.7),

$$
u(n) \leftrightarrow U(z) \Rightarrow nu(n) \leftrightarrow -z\frac{dU}{dz}
$$

and

$$
\begin{aligned}
\frac{z}{(z+1)} &\leftrightarrow (-1)^n 1(n-1) \\
\frac{d}{dz}\left[\frac{z}{z+1}\right] &= \frac{(z+1)-z}{(z+1)^2} = \frac{1}{(z+1)^2} \\
-\frac{z}{(z+1)^2} &\leftrightarrow n(-1)(-1)^n 1(-n-1) \\
-\frac{1}{3}\frac{z}{(z+1)^2} &\leftrightarrow \frac{n}{3}(-1)^n 1(-n-1) \\
\frac{4}{9}\frac{z}{z-2} &\leftrightarrow \frac{4}{9}(-1)(2)^n 1(-n-1)
\end{aligned}
$$

Substituting back the inverse of every term, we obtain,

$$
\begin{aligned}
g(n) &= \left[\frac{4}{9}(-1)^n 1(-n-1) + \frac{n}{3}(-1)^n 1(-n-1) - \frac{4}{9}2^n 1(-n-1)\right] \\
&= \left[\left(\frac{4}{9}+\frac{n}{3}\right)(-1)^n - \frac{4}{9}2^n\right] 1(-n-1)
\end{aligned}
$$