Department of Chemical Engineering
IIT Bombay
CL692, Digital Control
Solution to Assignment 1
To be completed by: 07 Aug 2006

1. Water flow problem:

$$A\frac{dh}{dt} = Q_i(t) - Q(t) = Q_i(t) - x(t)h(t)$$

Substituting

$$h = h_s + \Delta h, \quad Q_i = Q_{is} + \Delta Q_i \quad x = x_s + \Delta x$$

we obtain,

$$A\frac{d(h_s + \Delta h)}{dt} = Q_{is} + \Delta Q_i - (x_s + \Delta x)(h_s + \Delta h)$$

Subtracting the steady state equation

$$A\frac{dh_s}{dt} = Q_{is} - x_s h_s = 0$$

we obtain,

$$A\frac{d\Delta h}{dt} = \Delta Q_i - x_s \Delta h - h_s \Delta x$$

where, we have neglected the second order term. We obtain

$$\begin{aligned} \dot{\Delta h} &= \frac{\Delta Q_i}{A} - \frac{h_s \Delta x}{A} - \frac{x_s \Delta h}{A} \\ \therefore \quad \dot{\Delta h} &= \frac{-x_s \Delta k}{A} + \begin{bmatrix} \frac{1}{A} & \frac{-h_s}{A} \end{bmatrix} \begin{bmatrix} \Delta Q_i \\ \Delta x \end{bmatrix} \end{aligned}$$

2. ZOH with delay.

   (a) Using $\quad t = D + t_n - \tau$,

$$\begin{aligned} B_1 &= \int_{t_n}^{t_{n+D}} e^{F(t_{n+1} - \tau)} G d\tau = -\int_{D}^{0} e^{F(t_{n+1} - D - t_n + t)} G d\tau \\ &= e^{F(T_s - D)} \int_0^D e^{Ft} G d\tau \end{aligned}$$

   With the substitution $t = t_{n+1} - \tau$

$$\begin{aligned} B_0 &= \int_{t_{n+D}}^{t_{n+1}} e^{F(t_{n+1} - \tau)} G d\tau = -\int_{T_s - D}^{0} e^{Ft} G d\tau \\ &= \int_0^{T_s - D} e^{Ft} G d\tau \end{aligned}$$

Note that with slightly diferent substitution, we get different expressions for B_0 and B_1: with $t = \tau - t_n$, $\tau = t + t_n$, $d\tau = dt$,

$$\begin{aligned} I_1 &= \int_{t_n}^{t_n+D} e^{F(t_{n+1}-\tau)} G d\tau = \int_0^D e^{F(T_s-t)} G D\tau \\ &= e^{FT_s} \int_0^D e{-}FtGdt \triangleq B_1 \end{aligned}$$

with $t = \tau - (t_n + D)$,

$$\begin{aligned} I_2 &= \int_{t_{n+D}}^{t_{n+1}} e^{F(t_{n+1}-\tau} G d\tau = \int_0^{T_s-D} e^{F(t_{n+1}-t-t_n-D)} G d\tau \\ &= \int_0^{T_s-D} e^{F(T_s-D-t)} G dt = e^{F(T_s-D)} \int_0^{T_s-D} e^{-Ft} G dt \triangleq B_0 \end{aligned}$$

In either case, we get

$$x(n+1) = Ax(n) + B_1 u(n-1) + B_0 u(n)$$

(b) For $d = 1$, we had,

$$\begin{aligned} x(k+1) &= Ax(k) + B_0 u(k) + B_1 u(k-1) \\ \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} &= \begin{bmatrix} A & B_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} + \begin{bmatrix} B_0 \\ I \end{bmatrix} u(k) \end{aligned}$$

Notice that if $d = 2$,

$$x(k+1) = Ax(k) + B_1 u(k-2) + B_0 u(k-1)$$

The expressions for B_1 and B_0 do not change. The state space equivalent is

$$\begin{bmatrix} x(k+1) \\ u(k-1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & B_0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u(k)$$

(c) As mentioned above, B_0 and B_1 expressions are identical.

(d) For a general d, we get,

$$\begin{bmatrix} x(k+1) \\ u(k-d+1) \\ \vdots \\ u(k-1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & B_0 & 0 & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-d) \\ \vdots \\ u(k-2) \\ u(k-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u(k)$$