1. State Space Models

- Model is of the form $\dot{x} = Fx(t) + Gu(t)$
- x state
  - denotes variables that characterize the state of the system
  - knowing the state, know *everything* about the system
- $u(t)$ denotes the input to the system:
  - disturbance variable
  - manipulated or control variable
- In the flow system,
  - Inflow rate F_i is the disturbance variable
  - We could use the valve position (see the problem) as the manipulated variable or control effort

2. Constant Square Matrix - Exponential & Derivative

Need this to derive discrete models. Define

$$e^{Ft} \triangleq I + Ft + \frac{1}{2!}F^2t^2 + \frac{1}{3!}F^3t^3 + \dots$$

Example:

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$F^2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$F^3 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$e^{Ft} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & -t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -t^2 \\ 0 & t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & t^3 \\ 0 & -t^3 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 & t - \frac{t^2}{2} + \frac{t^3}{3!} + \cdots \\ 0 & 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Differentiate both sides:

$$\frac{d}{dt}\left(e^{Ft}\right) = \frac{d}{dt}\left(I + Ft + \frac{1}{2!}F^2t^2 + \frac{1}{3!}F^3t^3 + \dots\right)$$

$$= 0 + F + \frac{1}{2!}F^2 2t + \frac{1}{3!}F^3 3t^2 + \dots$$

$$= F + F^2 t + \frac{1}{2!}F^3t^2 + \dots$$

$$= (I + Ft + \frac{1}{2!}F^2t^2 + \dots)F = e^{Ft}F$$

In summary,

$$\frac{d}{dt}(e^{Ft}) = e^{Ft}F$$

3. Solution to State Space Model

Recall

$$e^{Ft} \stackrel{\triangle}{=} I + Ft + \frac{1}{2!}F^2t^2 + \dots$$
$$\frac{d}{dt}(e^{Ft}) = e^{Ft}F$$

Want to solve

$$\dot{x}(t) = Fx(t) + Gu(t)$$

Write this as follows:

$$\dot{x}(t) - Fx(t) = Gu(t)$$

Premultiply both sides by e^{-Ft} to get

$$e^{-Ft}\dot{x}(t) - e^{-Ft}Fx(t) = e^{-Ft}Gu(t)$$

This can be written as

$$\frac{d}{dt}(e^{-Ft}x(t)) = e^{-Ft}Gu(t)$$

Integrating both sides with respect to time from t_0 to t, we get

$$e^{-Ft}x(t) - e^{-Ft_0}x(t_0) = \int_{t_0}^{t} e^{-F\tau}Gu(\tau)d\tau$$
$$e^{-Ft}x(t) = e^{-Ft_0}x(t_0) + \int_{t_0}^{t} e^{-F\tau}Gu(\tau)d\tau$$

Premultiply by e^{Ft}:

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)d\tau$$

Interested in sampling instants only. Substitute (t_n, t_{n+1}) for (t_0, t).

$$x(t_{n+1}) = e^{F(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gu(\tau)d\tau$$

4. ZOH Equivalent of State Space Model

State space model:

$$\dot{x}(t) = Fx(t) + Gu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Recall solution to first equation:

$$x(t_{n+1}) = e^{F(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gu(\tau)d\tau$$

Assumption: piecewise constant u is used, *i.e.,*

$$u(\tau) = u(t_n), \quad t_n \le \tau < t_{n+1}$$

We will use only uniform sampling period of T_s:

$$T_s \stackrel{\triangle}{=} t_{n+1} - t_n$$

Solution to state space equation becomes,

$$x(t_{n+1}) = e^{FT_s}x(t_n) + \left[\int_{t_n}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gd\tau\right]u(t_n)$$

Define

$$A \stackrel{\triangle}{=} e^{FT_s}$$
$$B \stackrel{\triangle}{=} \int_{t_n}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gd\tau.$$
$$= \left[\int_0^{T_s} e^{F\tau}d\tau\right]G \quad \text{(Problem)}$$

A and B are constants.

$$x(t_{n+1}) = Ax(t_n) + Bu(t_n)$$

Constant $T_s \Rightarrow$ use sampling No.:

$$x(n+1) = Ax(n) + Bu(n)$$

Synchronize the sampling of $y(t)$ with that of $x(t)$ and $u(t)$:

$$x(n+1) = Ax(n) + Bu(n)$$
$$y(n) = Cx(n) + Du(n)$$

5. Example

$$\begin{aligned}\dot{x}(t) &= Fx(t) + Gu(t)\\ y(t) &= Cx(t) + Du(t)\\ A &= e^{FT_s}\\ B &= \left[\int_0^{T_s} e^{F\tau} d\tau\right] G\\ x(n+1) &= Ax(n) + Bu(n)\\ y(n) &= Cx(n) + Du(n)\end{aligned}$$

Example: Calculate the ZOH equivalent of the continuous system with $T_s = 1$ second

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ F &= \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}, \quad G = \begin{bmatrix}0\\1\end{bmatrix}\\ e^{Ft} &= \begin{bmatrix}1 & 1-e^{-t}\\0 & e^{-t}\end{bmatrix}\end{aligned}$$

$$A = e^{FT_s} = \begin{bmatrix}1 & 1-e^{-1}\\0 & e^{-1}\end{bmatrix} = \begin{bmatrix}1 & 0.632\\0 & 0.368\end{bmatrix}$$

Done three slides ago.

$$\begin{aligned}B &= \int_0^1 e^{F\tau} d\tau G\\ &= \int_0^1 \begin{bmatrix}1 & 1-e^{-\tau}\\0 & e^{-\tau}\end{bmatrix} d\tau \begin{bmatrix}0\\1\end{bmatrix}\\ &= \begin{bmatrix}\tau & \tau+e^{-\tau}\\0 & -e^{-\tau}\end{bmatrix}_0^1 \begin{bmatrix}0\\1\end{bmatrix}\\ &= \begin{bmatrix}\tau+e^{-\tau}\\-e^{-\tau}\end{bmatrix}_0^1 = \begin{bmatrix}1+e^{-1}-1\\-e^{-1}+1\end{bmatrix} = \begin{bmatrix}0.368\\0.632\end{bmatrix}\end{aligned}$$

Matlab Code:

```
F = [-1 0;1 0]; G = [1; 0];
C = [0 1]; D = 0; Ts=1;
sys = ss(F,G,C,D);
sysd = c2d(sys,Ts,'zoh')
```


6. ZOH Equivalent of System with Delay

Consider the model

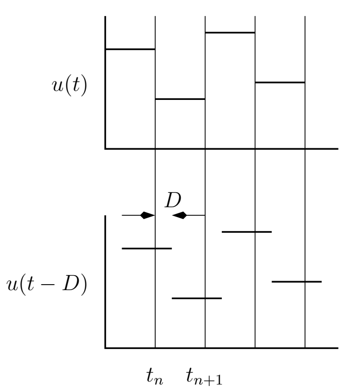
$$\dot{x}(t) = Fx(t) + Gu(t-D), \quad 0 \le D < T_s$$

Would like to arrive at

$$x(n+1) = Ax(n) + Bu(n),$$

the standard form.

Draw a typical u profile:



The solution of this becomes

$$x(t_{n+1}) = e^{F(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gu(\tau-D)d\tau$$

$u(\tau - D)$ - piecewise constant. Split the last term:

$$\begin{aligned}\int_{t_n}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gu(\tau-D)d\tau &= \int_{t_n}^{t_n+D} e^{F(t_{n+1}-\tau)}Gd\tau u(t_{n-1})\\ &\quad + \int_{t_n+D}^{t_{n+1}} e^{F(t_{n+1}-\tau)}Gd\tau u(t_n)\\ &= B_1u(t_{n-1}) + B_0u(t_n)\end{aligned}$$

Solution becomes

$$x(n+1) = Ax(n) + B_1u(n-1) + B_0u(n)$$

Can be written using an augmented state vector:

$$\begin{bmatrix}x(n+1)\\u(n)\end{bmatrix} = \begin{bmatrix}A & B_1\\0 & 0\end{bmatrix}\begin{bmatrix}x(n)\\u(n-1)\end{bmatrix} + \begin{bmatrix}B_0\\I\end{bmatrix}u(n)$$

In standard form. I is an identity matrix, same size as u.

7. Output

- It may not be possible to measure all the components of state
  - Expensive: measuring temperature in *all* trays of a tall distillation column.
  - There may not be sensors: eg. rate of change of viscosity.
- Only a subset/function of the state vector is often measured

Example: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

x_1 - height of liquid 1,
x_2 - height of liquid 2

- Measure only level 2:

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2$$

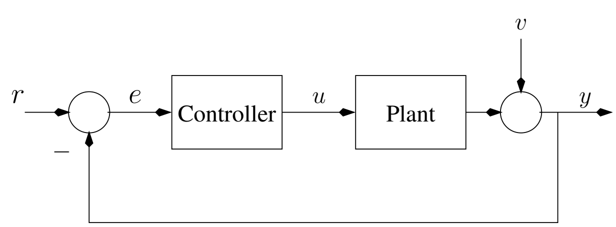
- Measure total level:

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2$$

- Modelled as

$$y = Cx + Du$$

8. Putting all together



- e is converted to digital signal using A/D converter
- u is made useful to real life system with D/A converter and ZOH
- Now, every block communicates with each other
- Plant model: $x(n+1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$
- Can convert this into transfer function to be taught
- Can also use state space approach, to be taught
- In any case, controller can understand the plant!
- How does controller work? What about intra sample behaviour?
- Can take care of it - will study. Use simulations to verify