1. A Procedure to Determine Impulse Response Coefficients

Consider an LTI system, $\xi(k)$ is white noise, u and ξ uncorrelated, *i.e.*, $r_{u\xi}(k) = 0$:

$$y(k) = \sum_{l=0}^{N} g(l)u(k-l) + \xi(k)$$

Multiplying both sides by $u(k-\tau)$ and summing for all k values over $[0, N]$,

$$\sum_{k=0}^{N} y(k)u(k-\tau) = \sum_{k=0}^{N}\sum_{l=0}^{N} g(l)u(k-l)u(k-\tau)$$

As u and ξ are uncorrelated, the second term on the right hand side is zero. We arrive at

$$r_{yu}(\tau) = \sum_{l=0}^{N} g(l)r_{uu}(\tau-l)$$

Evaluating this equation for different τ, and making use of $r_{uu}(n) = r_{uu}(-n)$,

$$\begin{bmatrix} r_{uu}(0) & \cdots & r_{uu}(N) \\ r_{uu}(-1) & \cdots & r_{uu}(N-1) \\ \vdots & & \\ r_{uu}(-N) & \cdots & r_{uu}(0) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix} = \begin{bmatrix} r_{yu}(0) \\ r_{yu}(1) \\ \vdots \\ r_{yu}(N) \end{bmatrix}.$$

Solve for g. Invertibility of this matrix is the *persistence of excitation* condition of u.

2. A Procedure to Determine Impulse Response Coefficients

$$\begin{bmatrix} r_{uu}(0) & \cdots & r_{uu}(N) \\ r_{uu}(-1) & \cdots & r_{uu}(N-1) \\ \vdots & & \\ r_{uu}(-N) & \cdots & r_{uu}(0) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix} = \begin{bmatrix} r_{yu}(0) \\ r_{yu}(1) \\ \vdots \\ r_{yu}(N) \end{bmatrix}$$

3 Unknowns: $$\begin{bmatrix} r_{uu}(0) & r_{uu}(1) & r_{uu}(2) \\ r_{uu}(1) & r_{uu}(0) & r_{uu}(1) \\ r_{uu}(2) & r_{uu}(1) & r_{uu}(0) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix} = \begin{bmatrix} r_{yu}(0) \\ r_{yu}(1) \\ r_{yu}(2) \end{bmatrix}$$

Recall the convolution model: $y(k) = \sum_{l=0}^{N} g(l)u(k-l) + \xi(k)$

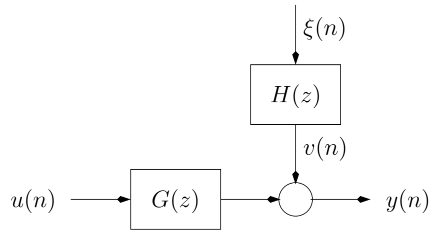
$$\begin{bmatrix} u(k) & u(k-1) & u(k-2) \\ u(k+1) & u(k) & u(k+1) \\ u(k+2) & u(k+1) & u(k) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix} = \begin{bmatrix} y(k) \\ y(k+1) \\ y(k+2) \end{bmatrix} - \begin{bmatrix} \xi(k) \\ \xi(k+1) \\ \xi(k+2) \end{bmatrix}$$

Of form $\Phi\theta = Z + E$. Premultiplying by transpose of coefficient matrix & ignoring noise,

$$\begin{bmatrix} u(k) & u(k+1) & u(k+2) \\ u(k-1) & u(k) & u(k+1) \\ u(k-2) & u(k-1) & u(k) \end{bmatrix} \begin{bmatrix} u(k) & u(k-1) & u(k-2) \\ u(k+1) & u(k) & u(k+1) \\ u(k+2) & u(k+1) & u(k) \end{bmatrix} = \begin{bmatrix} u(k) & u(k+1) & u(k+2) \\ u(k-1) & u(k) & u(k+1) \\ u(k-2) & u(k-1) & u(k) \end{bmatrix} \begin{bmatrix} y(k) \\ y(k+1) \\ y(k+2) \end{bmatrix}$$

$$\Phi^T\Phi\theta = \Phi^T Z$$

3. One Step Ahead Prediction Error Model



$$y(n) = G(z)u(n) + v(n)$$

Best estimate:

$$\hat{y}(n|n-1) = G(z)u(n) + \hat{v}(n|n-1)$$

Noise model, using white noise

$$v(n) = h(n) * \xi(n)$$

Can take leading term of h to be 1:

$$v(n) = \xi(n) + \sum_{l=1}^{\infty} h(l)\xi(n-l)$$

Best prediction of $v(n)$ is its expectation:

$$\begin{aligned}\hat{v}(n|n-1) &= \mathscr{E}\left[v(n)\right] \\ &= \mathscr{E}\left[\xi(n)\right] + \mathscr{E}\left[\sum_{l=1}^{\infty} h(l)\xi(n-l)\right]\end{aligned}$$

White noise, past terms

$$\hat{v}(n|n-1) = h(n) * \xi(n) - \xi(n)$$

In mixed notation:

$$\begin{aligned}\hat{v}(n|n-1) &= H(z)\xi(n) - \xi(n) = (H(z)-1)\xi(n) \\ &= (H(z)-1)H^{-1}(z)v(n) = (1-H^{-1}(z))v(n)\end{aligned}$$

Can show: H, H^{-1} stable. Substitute in $\hat{y}$:

$$\begin{aligned}\hat{y}(n|n-1) &= G(z)u(n) + (1-H^{-1}(z))v(n) \\ &= G(z)u(n) + [1-H^{-1}(z)][y(n)-G(z)u(n)] \\ &= H^{-1}(z)G(z)u(n) + [1-H^{-1}(z)]y(n)\end{aligned}$$

4. One Step Ahead PEM - Examples

Model:

$$y(k) = G(z)u(k) + H(z)\xi(k)$$

Prediction model:

$$\begin{aligned}\hat{y}(k|k-1) = {} & H^{-1}(z)G(z)u(k) \\ & + [1-H^{-1}(z)]y(k)\end{aligned}$$

FIR model:

$$y(k) = B(z)u(k) + \xi(k)$$

Obtain,

$$G(z) = B(z), \ H(z) = 1$$

Substituting, prediction model for FIR:

$$\hat{y}(k|k-1) = B(z)u(k)$$

Next, consider ARX model:

$$A(z)y(k) = B(z)u(k) + \xi(k)$$

Obtain,

$$G(z) = \frac{B(z)}{A(z)}, \ H(z) = \frac{1}{A(z)}$$

Substituting, prediction model for ARX:

$$\begin{aligned}\hat{y}(k|k-1) &= A(z)\frac{B(z)}{A(z)}u(k) + (1-A(z))y(k) \\ &= B(z)u(k) + (1-A(z))y(k)\end{aligned}$$

5. Models of Interest

- Finite Impulse Response (FIR) model, which is of the form,

$$y(n) = B(z)u(n) + \xi(n)$$

- Auto Regressive with eXogeneous input (ARX) model, which is of the form,

$$A(z)y(n) = B(z)u(n) + \xi(n)$$

- Auto Regressive Moving Average with eXogeneous (ARMAX) model, which is of the form,

$$A(z)y(n) = B(z)u(n) + C(z)\xi(n)$$

6. Models of Interest - Continued

- Auto Regressive Integrated Moving Average with eXogeneous (ARIMAX) model, which is of the form,

$$A(z)y(n) = B(z)u(n) + \frac{C(z)}{\Delta(z)}\xi(n)$$

  where, $\Delta = 1 - z^{-1}$.

- Output Error (OE) model, the general form of which is given as,

$$y(n) = G(z)u(n) + \xi(n)$$

  where, G is a transfer function. FIR is an OE model. Others are equation error models.

- Box Jenkins (BJ) model, which is of the form,

$$y(n) = G(z)u(n) + H(z)\xi(n)$$

  $G(z)$ and $H(z)$ are transfer functions

7. FIR Model as a Regression Equation

$$y(k) = \sum_{l=0}^{N} g(l)u(k-l) + \xi(k)$$

Writing the equations for $y(k)$, $y(k-1)$, $\ldots$ and stacking them one below another,

$$\begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \end{bmatrix} = \begin{bmatrix} u(k) & \cdots & u(k-N) \\ u(k-1) & \cdots & u(k-N-1) \\ \vdots & & \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix} + \begin{bmatrix} \xi(k) \\ \xi(k-1) \\ \vdots \end{bmatrix}$$

This is in the form of $Z(k) = \Phi(k)\theta + \Xi(k)$ with

$$Z(k) = \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \end{bmatrix}, \Phi(k) = \begin{bmatrix} u(k) & \cdots & u(k-N) \\ u(k-1) & \cdots & u(k-N-1) \\ \vdots & & \end{bmatrix} \theta = \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix}, \Xi(k) = \begin{bmatrix} \xi(k) \\ \xi(k-1) \\ \vdots \end{bmatrix}$$

Note that θ consists of the impulse response coefficients $g(0)$, $\ldots$, $g(N)$.

8. ARX Model as a Regression Equation

$$y(k) = a_1 y(k-1) + \sum_{l=0}^{N} g(l)u(k-l) + \xi(k)$$

$$\begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \end{bmatrix} = \begin{bmatrix} y(k-1) & u(k) & \cdots & u(k-N) \\ y(k-2) & u(k-1) & \cdots & u(k-N-1) \\ \vdots & & & \end{bmatrix} \begin{bmatrix} a_1 \\ g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix} + \begin{bmatrix} \xi(k) \\ \xi(k-1) \\ \vdots \end{bmatrix}$$

This is in the form of $Z(k) = \Phi(k)\theta + \Xi(k)$ with

$$Z(k) = \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \end{bmatrix}, \ \Phi(k) = \begin{bmatrix} y(k-1) & u(k) & \cdots & u(k-N) \\ y(k-2) & u(k-1) & \cdots & u(k-N-1) \\ \vdots & & & \end{bmatrix}, \ \theta = \begin{bmatrix} a_1 \\ g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix}$$

Note that θ consists of a_1 and the impulse response coefficients $g(0)$, $\ldots$, $g(N)$.

9. Least Squares Estimation: Regression Equation

- Least Squares Estimation is a convenient method to determine model parameters from experimental data.
- Let the model that relates the parameters and experimental data be given by

$$Z(k) = \Phi(k)\theta + \Xi(k).$$

- $Z(k)$ and $\Phi(k)$ consist of measurements and θ is a vector of parameters to be estimated.
- $\Xi(k)$ can be thought of as a mismatch between the best that the underlying model, characterized by θ, can predict and the actual measurement $Z(k)$. $\Xi(k)$ can also be thought of as random measurement noise.
- Known as the regression equation.
- Argument k is required in identification problems that received data on a continuous basis.
- If the problem at hand is to determine a set of parameters θ from one and only set of experimental data, there is no need to include this argument.

10. Solution to Least Squares Problem

Regression equation:

$$Z(k) = \Phi(k)\theta + \Xi(k)$$

Assume E to be negligible. Model:

$$\hat{Z}(k) = \Phi(k)\hat{\theta}(k)$$

$\hat{\theta}(k)$: estimate. Error:

$$\tilde{Z}(k) \stackrel{\triangle}{=} Z(k) - \hat{Z}(k)$$

Want $\tilde{Z}$ to be small. 2×2 example:

$$Z(k) = \begin{bmatrix} y(k) \\ y(k-1) \end{bmatrix}, \hat{Z}(k) = \begin{bmatrix} \hat{y}(k) \\ \hat{y}(k-1) \end{bmatrix}$$

$$\tilde{Z}(k) = \begin{bmatrix} \tilde{z}(k) = y(k) - \hat{y}(k) \\ \tilde{z}(k-1) = y(k-1) - \hat{y}(k-1) \end{bmatrix}$$

Form an objective function to minimize:

$$\begin{aligned} \tilde{Z}^T(k)W(k)\tilde{Z}(k) &= \begin{bmatrix} \tilde{z}(k) & \tilde{z}(k-1) \end{bmatrix} \\ &\quad \begin{bmatrix} w(k) & 0 \\ 0 & w(k-1) \end{bmatrix} \begin{bmatrix} \tilde{z}(k) \\ \tilde{z}(k-1) \end{bmatrix} \\ &= \begin{bmatrix} \tilde{z}(k) & \tilde{z}(k-1) \end{bmatrix} \begin{bmatrix} w(k)\tilde{z}(k) \\ w(k-1)\tilde{z}(k-1) \end{bmatrix} \\ &= w(k)\tilde{z}^2(k) + w(k-1)\tilde{z}^2(k-1) \end{aligned}$$

Minimize objective function to find $\hat{\theta}$:

$$\begin{aligned} J[\hat{\theta}(k)] &= w(k)\tilde{z}^2(k) + \cdots + w(k-N)\tilde{z}^2(k-N) \\ &= \tilde{Z}(k)W(k)\tilde{Z}(k) \\ &= [Z(k) - \hat{Z}(k)]^T W(k)[Z(k) - \hat{Z}(k)] \end{aligned}$$

Minimize J and determine $\hat{\theta}_{\rm WLS}$:

$$\hat{\theta}_{\rm WLS}(k) = \text{arg} \min_{\theta} J[\hat{\theta}(k)]$$

11. Solution to Least Squares Problem - Continued

Recall $\hat{\theta}_{\text{WLS}}$ is obtained by minimizing

$$J[\hat{\theta}(k)] = [Z(k) - \hat{Z}(k)]^T W(k)[Z(k) - \hat{Z}(k)]$$
$$\hat{Z}(k) = \Phi(k)\hat{\theta}(k)$$

Substituting for $\hat{Z}(k)$,

$$J[\hat{\theta}(k)] = [Z(k) - \Phi(k)\hat{\theta}(k)]^T W(k)[Z(k) - \Phi(k)\hat{\theta}(k)]$$

We drop the argument k temporarily for convenience and obtain,

$$J[\hat{\theta}] = Z^T W Z - 2Z^T W\Phi\hat{\theta} + \hat{\theta}^T\Phi^T W\Phi\hat{\theta}$$

To find $\hat{\theta}$ at which J is minimum, differentiate and equate to zero:

$$\frac{\partial J}{\partial\hat{\theta}} = -2\Phi^T W Z + 2\Phi^T W\Phi\hat{\theta} = 0$$

From this, we arrive at the normal equation,

$$\Phi^T W\Phi\hat{\theta} = \Phi^T W Z$$

Assume that $\Phi^T W\Phi$ is nonsingular. Persistence Condition.

$$\hat{\theta}_{\text{WLS}}(k) = [\Phi^T(k)W(k)\Phi(k)]^{-1}\Phi^T(k)W(k)Z(k)$$