1. State Space Approach to Pole Placement Controller

Start with our state space model:

$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1).$$

- Let u be a scalar, i.e. only one control effort.
- Suppose all the states are measured.
- Can we use a state feedback controller to get a desired closed loop characteristic polynomial?

Take Z-transform:

$$zX(z) = AX(z) + bU(z) + x_0 z$$

The first term is:

$$(zI - A)X(z) = bU(z)$$

$$X(z) = (zI - A)^{-1}bU(z) = \frac{\mathrm{adj}(zI - A)}{|zI - A|}bU(z)$$

2. Eigenvalues = Poles

- Poles of transfer function = eigen values of A matrix.
- Let λ denote eigenvalues and v eigenvectors:

$$Av = \lambda v,$$

$$|\lambda I - A| = 0$$

Roots of $|zI - A|$ gives eigenvalues of A matrix. Example:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$(zI - A) = \begin{bmatrix} z-1 & -2 \\ 0 & z-3 \end{bmatrix}$$

$$(zI - A)^{-1} = \frac{1}{(z-1)(z-3)} \begin{bmatrix} z-3 & 2 \\ 0 & z-1 \end{bmatrix}$$

- 1 and 3 are poles of transfer function - also eigenvalues of A

3. State Space Approach to Pole Placement Controller

Recall our requirement and ss model:

$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1).$$

Can we use a state feedback controller to get a desired closed loop characteristic polynomial?

$$u(m) = -Kx(m) + v(m) = -\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1(m) \\ x_2(m) \\ \vdots \\ x_n(m) \end{bmatrix} + v(m$$

Substituting in state equation,

$$\begin{aligned} x(m+1) &= Ax(m) + b[-Kx(m) + v(m)] + x_0\delta(m+1) \\ &= (A - bK)x(m) + bv(m) + x_0\delta(m+1) \end{aligned}$$

A suitable K could make the system matrix behave better.

4. SS - Pole Placement Controller - Example

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Eigenvalues of A are 1, 3. Unstable.

$$\begin{aligned} A - bK &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -K_1 & 3 - K_2 \end{bmatrix} \end{aligned}$$

If $K_1 = 0.5$, $K_2 = 3.5$,

$$A - bK = \begin{bmatrix} 1 & 2 \\ -0.5 & -0.5 \end{bmatrix}.$$

- Eigenvalues are $0.25 \pm 0.6614j$
- Magnitude = 0.7071, stable

5. K Should Produce Desirable Characteristic Equation

Start with state space equation:

$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1).$$

Use controller,

$$u(m) = -Kx(m) + v(m)$$

Substitute in state space equation:

$$x(m+1) = (A - bK)x(m) + bv(m) + x_0\delta(m+1)$$

Taking Z-transforms,

$$[zI - (A - bK)]X(z) = bV(z) + x_0 z$$

$$\begin{aligned} X(z) &= [zI - (A - bK)]^{-1}(bV(z) + x_0 z) \\ &= \frac{\mathrm{adj}[zI - (A - bK)]}{|zI - (A - bK)|}(bV(z) + x_0 z) \end{aligned}$$

6. State Space Approach to Pole Placement Controller

$|zI - (A - bK)| = 0$ is the closed loop characteristic equation. We may want the characteristic polynomial to be equal to

$$\begin{aligned} \alpha_c(z) &= z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n \\ &= (z - \beta_1)(z - \beta_2)\cdots(z - \beta_n) \end{aligned}$$

- How does one find K to satisfy the above requirement?
- Can one always find such a K?
- Can equate the coefficients and solve for K. But not an easy problem.
- We will show that if the pair (A, b) is in controller canonical form, to be defined next, K can be easily determined.

7. Controller Canonical form

- (A, b) - controller canonical form if A and b are as in

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Can show that (Problem 14.6)
  $|zI - A| = z^n + a_1 z^{n-1} + \cdots + a_n$
- With $u = Kx + v$, we obtained
  $x(m+1) = (A - bK)x(m) + bv(m) + x_0\delta(m+1)$

8. State Space Approach to Pole Placement Controller

$A - BK =$

$$\begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \cdots & -a_{n-2} - K_{n-2} & -a_{n-1} - K_{n-1} & -a_n - K_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \varphi(z) &= |zI - (A - bK)| \\ &= z^n + (a_1 + K_1)z^{n-1} + \cdots + (a_{n-1} + K_{n-1})z + (a_n + K_n) \end{aligned}$$

- If the desired characteristic polynomial is
  $\alpha_c(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n$,
- Equating the coefficients of powers of z in $\varphi(z)$ and $\alpha_c(z)$, can determine $\begin{bmatrix} K_1 & \cdots & K_n \end{bmatrix} = \begin{bmatrix} \alpha_1 - a_1 & \cdots & \alpha_n - a_n \end{bmatrix}$

9. Ackermann's formula

Suppose that A and b are in controller canonical form.
Then the characteristic equation of A is given by

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

As per Cayley Hamilton's theorem, a matrix satisfies its own characteristic equation:

$$A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0$$

Evaluate the desired characteristic polynomial

$$\alpha_c(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n$$

at A to arrive at

$$\begin{aligned}\alpha_c(A) &= A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I \\ &= (\alpha_1 - a_1) A^{n-1} + \cdots + (\alpha_{n-1} - a_{n-1}) A + (\alpha_n - a_n) I\end{aligned}$$

10. Ackermann's Formula

Recall controller canonical form:

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Pre-multiply powers of A by e_n:

$$\begin{aligned} e_n^T A &= e_{n-1}^T \\ e_n^T A^2 &= (e_n^T A) A = e_{n-1}^T A = e_{n-2}^T \\ &\vdots \\ e_n^T A^{n-1} &= e_1^T \end{aligned}$$

11. Ackermann's Formula - Continued

Recall

$$\alpha_c(A) = (\alpha_1 - a_1)A^{n-1} + \cdots + (\alpha_{n-1} - a_{n-1})A + (\alpha_n - a_n)I$$

Pre-multiplying e_n^T

$$\begin{aligned} e_n^T\alpha_c(A) &= (\alpha_1 - a_1)e_n^T A^{n-1} + \cdots + (\alpha_{n-1} - a_{n-1})e_n^T A \\ &+ (\alpha_n - a_n)e_n^T \\ &= (\alpha_1 - a_1)e_1^T + \cdots + (\alpha_{n-1} - a_{n-1})e_{n-1}^T \\ &+ (\alpha_n - a_n)e_n^T \\ &= \begin{bmatrix}\alpha_1 - a_1 & \cdots & \alpha_{n-1} - a_{n-1} & \alpha_n - a_n\end{bmatrix} = K \end{aligned}$$

Thus the pole placing controller is given by

$$K = e_n^T\alpha_c(A) = \text{last row of } \alpha_c(A)$$

12. What if A and b are not in Controller Canonical Form

- (A, b) not in controller canonical form
- $\exists S$ such that $S^{-1}AS = \overline{A}$, $S^{-1}b = \bar{b}$, $(\overline{A}, \bar{b})$ controllable, iff $\mathcal{C} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b\end{bmatrix}$ is nonsingular - assume this
- Define new state $\overline{x}$: $x = S\overline{x}$. S is nonsingular and constant

State space equation $x(m+1) = Ax(m) + bu(m)$ becomes,

$$\begin{aligned} S\overline{x}(m+1) &= AS\overline{x}(m) + bu(m) \\ \overline{x}(m+1) &= S^{-1}AS\overline{x}(m) + S^{-1}bu(m) \end{aligned}$$

Define new variables,

$$\overline{A} \triangleq S^{-1}AS, \quad \bar{b} \triangleq S^{-1}b$$

State space equation in new coordinates (u not changed):

$$\overline{x}(m+1) = \overline{A}\overline{x}(m) + \bar{b}u(m)$$

13. Ackermann's Formula

- New state $\overline{x}$: $x = S\overline{x}$
- $\overline{A} = \overline{S}^{-1}AS$, $\overline{b} = S^{-1}b$
- $\overline{A}$, $\overline{b}$ are in controller canonical form
- Can design $\overline{K}$ by equating coefficients:

$$u(m) = -\overline{K}\overline{x}(m) + v(m) = -\overline{K}S^{-1}x(m) + v(m)$$

In original coordinates, controller: $K = \overline{K}S^{-1}$. Substitute in

$$\begin{aligned}
\overline{x}(m+1) &= \overline{A}\overline{x}(m) + \overline{b}u(m) \\
\overline{x}(m+1) &= \overline{A}\overline{x}(m) - \overline{b}\overline{K}\overline{x}(m) + bv(m) \\
&= (\overline{A} - \overline{bK})\overline{x}(m) + bv(m)
\end{aligned}$$

How useful is this characteristic polynomial?

14. Ackermann's Formula

Characteristic polynomial in new coordinates:

$$\overline{\varphi}(z) = |zI - (\overline{A} - \overline{bK})|$$

Substitute $I = S^{-1}S$, $\overline{A} = \overline{S}^{-1}AS$, $\overline{b} = S^{-1}b$

$$= \left|zS^{-1}S - \left(S^{-1}AS - S^{-1}bKS\right)\right|$$

Pull out S^{-1} from left and S from right:

$$= |S^{-1}\left[zI - (A - bK)\right]S|$$

Determinant of products is product of determinants:

$$= |S^{-1}S||zI - (A - bK)| = \varphi(z)$$

Ch. polynomial of original and transformed system are the same.
Controller: $\overline{K} = e_n^T\alpha_c(\overline{A})$.

15. Evaluating Controller Expression Directly

Recall

$$
\begin{aligned}
x &= S\overline{x}, \quad \overline{A} = S^{-1}AS, \quad \overline{b} = S^{-1}b \\
K &= \overline{K}S^{-1} \\
\overline{K} &= e_n^T \alpha_c(\overline{A})
\end{aligned}
$$

Evaluate powers of A:

$$
\begin{aligned}
\overline{A}^2 &= S^{-1}ASS^{-1}AS = S^{-1}A^2S \\
\overline{A}^3 &= \overline{A}^2\overline{A} = S^{-1}A^2SS^{-1}AS = S^{-1}A^3S \\
\overline{Ab} &= S^{-1}Ab, \quad \overline{A}^2\overline{b} = S^{-1}A^2b, \ \cdots \ \overline{A}^{n-1}\overline{b} = S^{-1}A^{n-1}b \\
\overline{\mathcal{C}} &= \left[\overline{b} \ \overline{Ab} \ \cdots \ \overline{A}^{n-1}\overline{b}\right] = S^{-1}\left[b \ \ Ab \ \cdots \ A^{n-1}b\right] \\
&= S^{-1}\mathcal{C}
\end{aligned}
$$

16. Evaluating Controller Expression Directly

Recall

$$\overline{\mathcal{C}} = S^{-1}\mathcal{C}$$

Therefore, $\mathcal{C}$ has full rank, if and only if $\overline{\mathcal{C}}$ is of full rank.

$$\overline{K} = e_n^T\alpha_c(\overline{A}) = e_n^T S^{-1}\alpha_c(A)S$$

This is because,

$$\overline{A}^n = S^{-1}A^nS$$

Let us Calculate K:

$$K = \overline{K}S^{-1} = \overline{e}_n^T S^{-1}\alpha_c(A) = e_n^T\overline{\mathcal{C}}\mathcal{C}^{-1}\alpha_c(A) = e_n^T\mathcal{C}^{-1}\alpha_c(A)$$

Ackermann's formula