1. Use ACF to Determine Order of AR Process?

Calculate the ACF of AR(1) process:

$$y(k) + a_1 y(k-1) = \xi(k)$$

Multiplying by $y(k-1)$, $y(k-2)$, ..., $y(k-l)$ and take expectation:

$$\begin{aligned} \gamma_{yy}(1) + a_1\gamma_{yy}(0) &= 0 \\ \gamma_{yy}(2) + a_1\gamma_{yy}(1) &= 0 \\ &\vdots \\ \gamma_{yy}(l) + a_1\gamma_{yy}(l-1) &= 0 \end{aligned}$$

$$\gamma_{yy}(l) = -a_1\gamma_{yy}(l-1) = -a_1(-a_1\gamma_{yy}(l-2)) = \ldots = (-1)^l a_1^l \gamma_{yy}(0)$$

- Therefore, $\rho_{yy}(l) = (-1)^l a_1^l$. Thus, ACF never dies out for an AR process and hence cannot be used for detecting the order of an AR process.
- Although there is no direct correlation between $y(k)$ and $y(k-l)$ for $l > 1$, it appears to exist due to auto-regression.

2. PACF Approach

To determine the order of AR(p) process:

$$y(n) + a_1 y(n-1) + \cdots + a_p y(n-p) = \xi(n)$$

1. Let $j = 1$
2. Assume that the system is an AR(j) model:

$$y(n) + a_{1j} y(n-1) + \cdots + a_{jj} y(n-j) = \xi(n)$$

3. Multiplying this equation by $y(n-k)$ and taking expectation, we obtain

$$\gamma_{yy}(k) + a_{1j}\gamma_{yy}(k-1) + \cdots + a_{jj}\gamma_{yy}(k-j) = 0, \quad \forall k \geq 1$$

4. Write this for $k = 1$ to j, arrive at j equations. Solve them for a_{jj}.
5. If $j < j_{max}$, increment j by 1 and go to step 2 above.

Note that j_{max} should be chosen to be greater than the expected p. A plot of a_{jj} *vs.* j will have a cut off from $j = p + 1$ onwards.

3. Determination of AR(2) by PACF Approach

Recall the basic equation:

$$y(n) + a_{1j}y(n-1) + \cdots + a_{jj}y(n-j) = \xi(n) \tag{1}$$

Determine the order of

$$y(n) - y(n-1) + 0.5y(n-2) = \xi(n) \tag{2}$$

For $j = 1$, Eq. 1 becomes

$$\gamma_{yy}(k) + a_{11}\gamma_{yy}(k-1) = 0, \ \forall k \geq 1.$$

For $k = 1$, the above equation becomes

$$\begin{aligned} \gamma_{yy}(1) + a_{11}\gamma_{yy}(0) &= 0 \\ a_{11} &= -\frac{\gamma_{yy}(1)}{\gamma_{yy}(0)} \end{aligned} \tag{3}$$

For $j = 2$, Eq. 1 becomes

$$\gamma_{yy}(k) + a_{12}\gamma_{yy}(k-1) + a_{22}\gamma_{yy}(k-2) = 0$$

4. Determination of AR(2) by PACF Approach - Ctd - 1

$$\gamma_{yy}(k) + a_{12}\gamma_{yy}(k-1) + a_{22}\gamma_{yy}(k-2) = 0$$

$k \geq 1$. For $k = 1, \ 2$, this equation becomes,

$$\begin{bmatrix} \gamma_{yy}(0) & \gamma_{yy}(1) \\ \gamma_{yy}(1) & \gamma_{yy}(0) \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = - \begin{bmatrix} \gamma_{yy}(1) \\ \gamma_{yy}(2) \end{bmatrix}. \tag{4}$$

For $j = 3$, Eq. 1 becomes

$$\gamma_{yy}(k) + a_{13}\gamma_{yy}(k-1) + a_{23}\gamma_{yy}(k-2) + a_{33}\gamma_{yy}(k-3) = 0$$

$\forall k \geq 1$. For $k = 1, 2, 3$, it becomes,

$$\begin{bmatrix} \gamma_{yy}(0) & \gamma_{yy}(1) & \gamma_{yy}(2) \\ \gamma_{yy}(1) & \gamma_{yy}(0) & \gamma_{yy}(1) \\ \gamma_{yy}(2) & \gamma_{yy}(1) & \gamma_{yy}(0) \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = - \begin{bmatrix} \gamma_{yy}(1) \\ \gamma_{yy}(2) \\ \gamma_{yy}(3) \end{bmatrix} \tag{5}$$

To solve these equations for a_{jj}, we need to calculate $\gamma_{yy}(k)$, $k = 0$ to 3.

5. Determination of AR(2) by PACF Approach - Ctd - 2

Determine the order of

$$y(n) - y(n-1) + 0.5y(n-2) = \xi(n) \tag{6}$$

Multiply by $\xi(n)$, take expectation:

$$\gamma_{y\xi}(0) = \gamma_{ee}(0) = \sigma_\xi^2$$

Multiply Eq. 6 by $y(n)$, $y(n-1)$ and $y(n-2)$, one at a time, take expectation:

$$\begin{bmatrix} 1 & -1 & 0.5 \\ -1 & 1.5 & 0 \\ 0.5 & -1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{yy}(0) \\ \gamma_{yy}(1) \\ \gamma_{yy}(2) \end{bmatrix} = \sigma_\xi^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving it, we obtain

$$\begin{bmatrix} \gamma_{yy}(0) \\ \gamma_{yy}(1) \\ \gamma_{yy}(2) \end{bmatrix} = \begin{bmatrix} 2.4 \\ 1.6 \\ 0.4 \end{bmatrix} \sigma_\xi^2 \tag{7}$$

Multiply Eq. 6 by $y(n-3)$ and take expectation

$$\gamma_{yy}(3) - \gamma_{yy}(2) + 0.5\gamma_{yy}(1) = 0$$

solving which, we obtain

$$\gamma_{yy}(3) = -0.4\sigma_\xi^2 \tag{8}$$

Substitute 7 to 8 in 3 - 5 and determine a_{ij}. We obtain

$$a_{11} = -0.67$$
$$a_{22} = 0.5$$
$$a_{33} = 0$$

As expected for the AR(2) process, $a_{jj} = 0$ for $j > 2$.

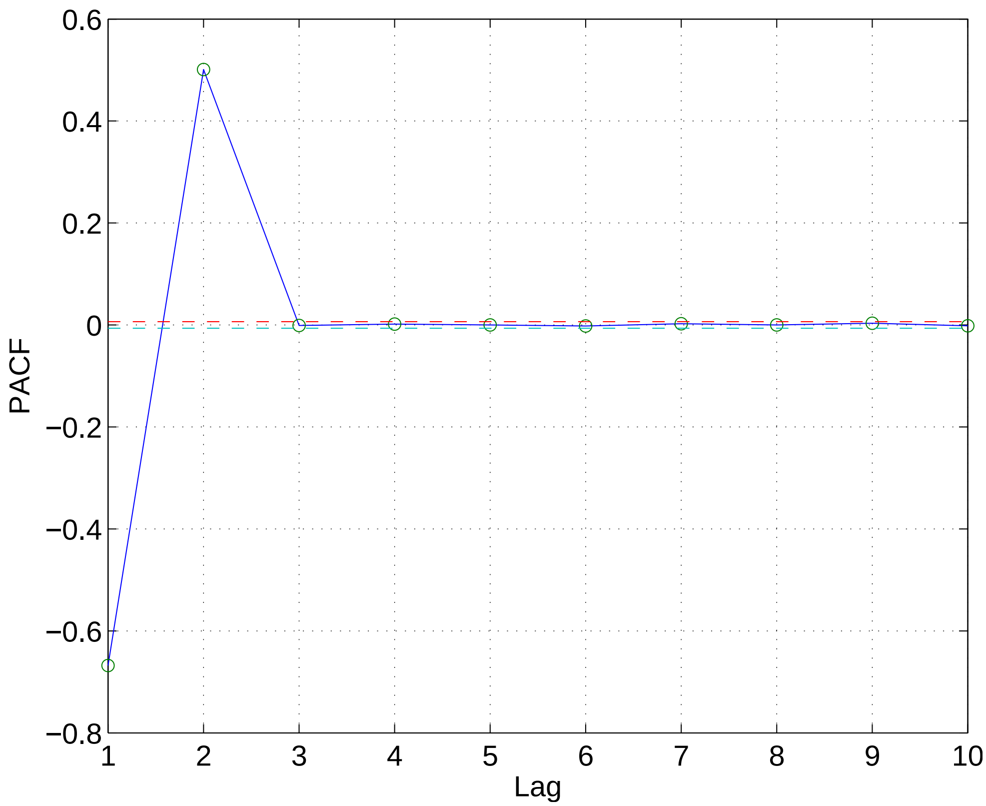
6. pacf_ex.m

```
% Define model and generate data
  m = idpoly([1,-1,0.5],[],1);
  e = 0.1*randn(100000,1);
  y = sim(m,e);

% Plot noise and plant output
  subplot(2,1,1), plot(y(1:500))
  title('Plant output and noise input vs. time',...
        'FontSize',14)
  ylabel('Plant output y','FontSize',14)
  subplot(2,1,2), plot(e(1:500))
  ylabel('Noise input e','FontSize',14)
  xlabel('Sampling instant, k','FontSize',14)

% Generate PACF and plot
  figure, pacf(y,10);
```

7. pacf.m

```
function [ajj] = pacf(y,M)
ryy = xcorr(y,'coeff');
len = length(ryy);
zero = (len+1)/2;
ryy0 = ryy(zero);
ryy_one_side = ryy(zero+1:len);
ajj = [];
for j = 1:M,
  ajj = [ajj pacf_mat(ryy0,ryy_one_side,j,1)];
end
p = 1:length(ajj);
N = length(p);
lim = 2/sqrt(length(y));

% Plot the figure

A = axes('FontSize',14);
set(get(A,'Xlabel'),'FontSize',14);
plot(p,ajj,p,ajj,'o',p,lim*ones(N,1),'--',...
                p,-lim*ones(N,1),'--')
ylabel('PACF'),xlabel('Lag'), grid
```


8. Partial Auto Correlation Function



9. Determination of MA(q) and AR(p) Processes

MA(q) process:

$$y(n) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q)$$

- A plot of $\{\gamma_{yy}(k)\}$ *vs.* k becomes zero for all $k > q$.

Known as ACF plot

AR(p) process:

$$y(k) + a_1y(k-1) + \cdots + a_py(k-p) = \xi(k)$$

- Assume $p = 1$, calculate a_1, call it a_{11}
- Assume $p = 2$, calculate a_2, call it a_{22}
- Repeat this enough number of times
- Plot a_{11}, a_{22}, ..., a_{jj} *vs.* j
- From $j = p + 1$ onwards, $a_{jj} = 0$

Known as PACF plot

10. Determination of ARMA(p, q) Process by Trial and Error

$$y(n) + a_1y(n-1) + \cdots + a_py(n-p) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q)$$

1. Plot the ACF and PACF to check if it is a pure AR, MA or a mixed process.
2. For mixed process, start with an AR(1) model (use the arma function in MATLAB).
3. Compute the residuals of this model (use the pe function in MATLAB).
4. Examine the ACF and PACF of the residuals.
5. If a mixed process is observed, estimate an ARMA(1,1) model.
6. Implement steps 3 and 4. Increase the AR and MA order alternately until convergence.

In practice, commonly occurring stochastic processes can be adequately represented by an arma(2,2) or by a lower order process

11. Determination of ARMA(p, q) Process by Trial and Error

$$y(n) = \frac{1 - 0.3z^{-1}}{1 - 0.8z^{-1}}\xi(n)$$

```
% Set up the model for simulation
arma_mod = idpoly(1,0,[1 -0.3],[1 -0.8],1,1);

% Generate the inputs for simulation
% Deterministic Input can be anything
u = zeros(2048,1);
e = randn(2048,1);

% Simulate the model
y = idsim(arma_mod,[u e]);

% Plot ACF and PACF for 10 lags
figure, plotacf(y,1e-03,11,1);
figure, pacf(y,10);

% Estimate AR(1) model and present it
mod_est1 = armax(y,[1 0]); present(mod_est1)

% compute the residuals
```

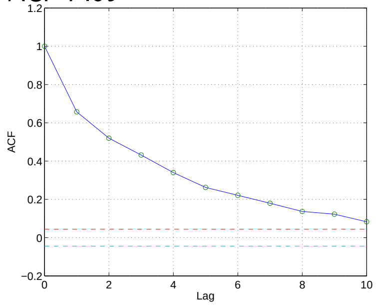


```
err_mod1 = pe(mod_est1,y);

% Plot ACF and PACF for 10 lags
figure
plotacf(err_mod1,1e-03,11,1);
figure, pacf(err_mod1,10);

% Check ACF and PACF of residuals
mod_est2 = armax(y,[1 1]); present(mod_est2)
err_mod2 = pe(mod_est2,y);

% Plot ACF and PACF for 10 lags
figure
plotacf(err_mod2,1e-03,11,1);
figure, pacf(err_mod2,10);
```

12. ACF and PACF Plots

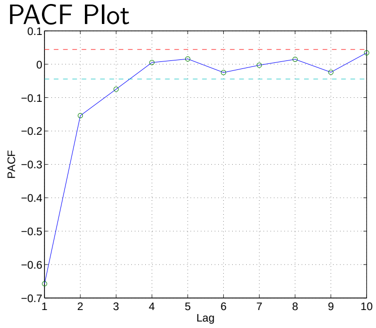
Original Process:

ACF Plot



Slow decay $\Rightarrow$ AR Process

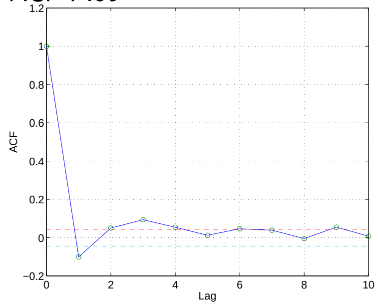
PACF Plot



Many nonzero coefficients $\Rightarrow$ Mixed Process

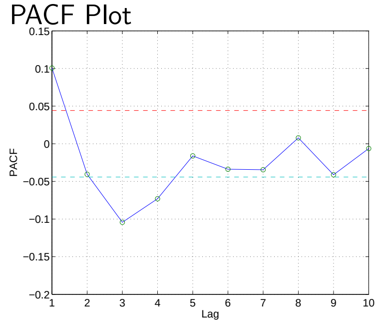
Residual of AR(1) Process (24):

ACF Plot



Good, but can improve

PACF Plot

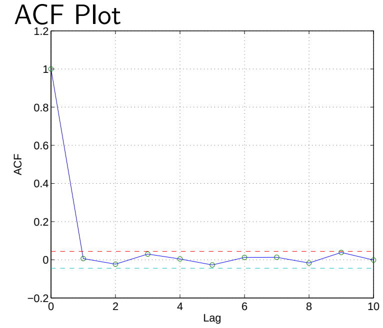


Good, but can improve

Try ARMA(1,1) process

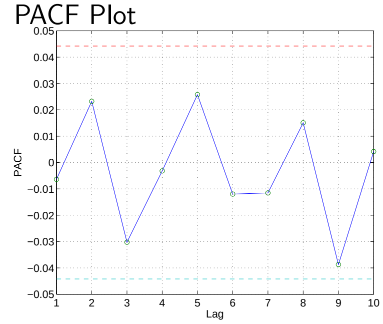
Residual of ARMA(1,1) Process (33):

ACF Plot



Behaves like white noise

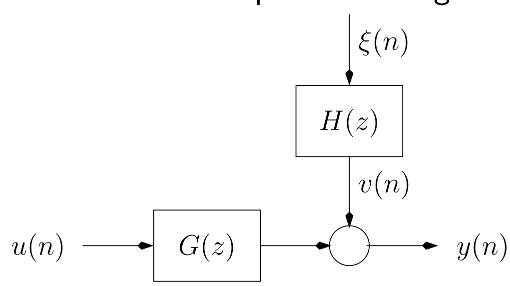
PACF Plot



White noise is confirmed

13. Estimation of Impulse Response

Consider usual plant configuration:



$$y(m) = g(m) * u(m) + h(m) * \xi(m)$$

Convolve with $u(-m)$. As u and e are uncorrelated ($r_{ue}(n) = 0$),

$$r_{yu}(m) = g(m) * r_{uu}(m),$$

where, r denotes correlation. Taking Z-transform,

$$\Phi_{yu}(z) = G(z)\Phi_{uu}(z)$$

Taking Fourier Transform,

$$\Phi_{yu}\left(e^{j\omega}\right) = G\left(e^{j\omega}\right)\Phi_{uu}\left(e^{j\omega}\right)$$

As $r_{uu}(n)$ is real and even, Φ_{uu} real. If $\Phi_{uu}\left(e^{j\omega}\right) = K$, a constant,

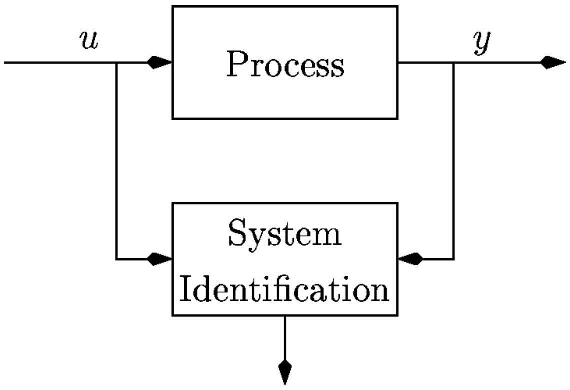
$$G\left(e^{j\omega}\right) = \frac{1}{K}\Phi_{yu}\left(e^{j\omega}\right).$$

On inverting this, we obtain

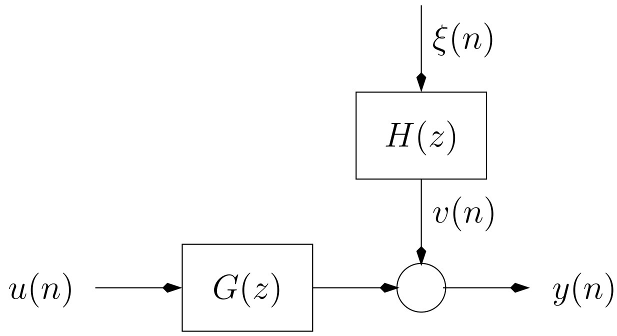
$$\{g(n)\} = \frac{1}{K}\{r_{yu}(n)\}.$$

- This method of estimating impulse response reduces noise: summation is a smoothing operation.
- If input were white, $\Phi_{uu}\left(e^{j\omega}\right) = K$, a constant

System Identification Problem:



Want to find $G(z)$ and $H(z)$:



First find G only, assuming white noise: