1. Area Based Approximation - Summary

Trapezoidal approximation:

$$\frac{1}{s} \leftrightarrow \frac{T_s}{2}\frac{z+1}{z-1}$$

Backward difference:

$$\frac{1}{s} \leftrightarrow T_s\frac{z}{z-1}$$

Forward difference:

$$\frac{1}{s} \leftrightarrow \frac{T_s}{z-1}$$

2. PID Controller - Basic Design

Basic version:

$$u(t) = K\left[e(t) + \frac{1}{\tau_i}\int_0^t e(t)dt + \tau_d\frac{de(t)}{dt}\right]$$

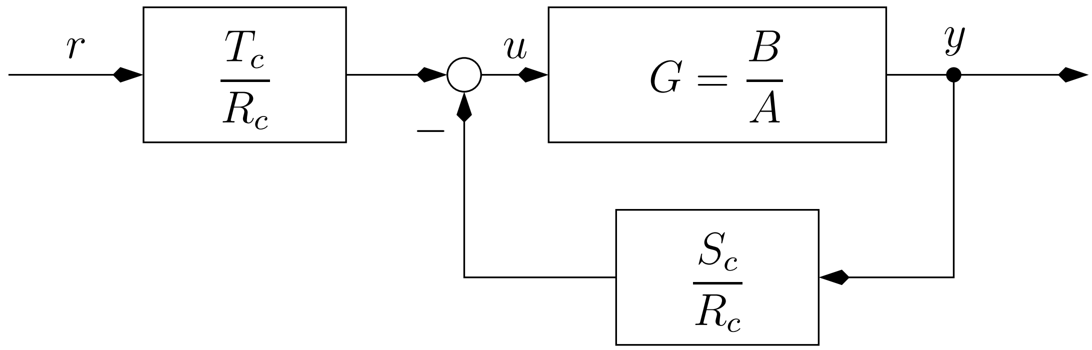
$$U(s) = K(1 + \frac{1}{\tau_i s} + \tau_d s)E(s)$$

Filtered derivative mode:

$$u(t) = K\left(1 + \frac{1}{\tau_i s} + \frac{\tau_d s}{1 + \frac{\tau_d s}{N}}\right)e(t)$$

N is a large number, of the order of 10

3. Offset-Free Tracking of Steps with Integral Action



- Let an integral action be present in the loop
- One of the following conditions should be met for offset free tracking:

$$T_c = S_c$$
$$T_c = S_c(1)$$
$$T_c(1) = S_c(1)$$

4. $S_c = T_c$: Filtered form of PID control

$$u(t) = K\left(1 + \frac{1}{\tau_i s} + \frac{\tau_d s}{1 + \frac{\tau_d s}{N}}\right) e(t) = \frac{S_c}{R_c} e(t)$$

$$s \leftrightarrow \frac{1 - z^{-1}}{T_s}$$

$$\frac{S_c}{R_c} = K\left[1 + \frac{T_s}{\tau_i}\frac{1}{1 - z^{-1}} - \frac{N r_1 (1 - z^{-1})}{1 + r_1 z^{-1}}\right]$$

$$R_c(z) = (1 - z^{-1})(1 + r_1 z^{-1})$$

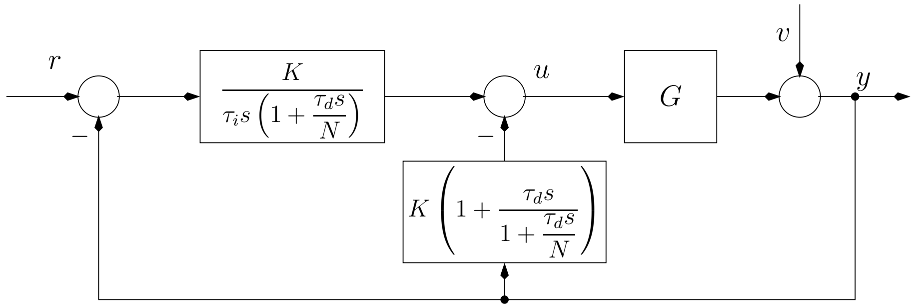
$$S_c(z) = s_0 + s_1 + s_2 z^{-2}$$

Unknowns are:

$$r_1 = -\frac{\tau_d/N}{\tau_d/N + T_s}, \quad s_0 = K\left(1 + \frac{T_s}{\tau_i} - N r_1\right)$$

$$s_1 = K\left[r_1\left(1 + \frac{T_s}{\tau_i} + 2N\right) - 1\right], \quad s_2 = -K r_1 (1 + N)$$

5. $T_c = S_c(s=0)$: r Not Sent through Derivative Mode



$$u(t) = \frac{K}{\tau_i s\left(1+\frac{\tau_d s}{N}\right)}(r(t) - y(t)) - K\left[1 + \frac{\tau_d s}{1+\frac{\tau_d s}{N}}\right] y(t)$$

$$= \frac{K}{\tau_i s\left(1+\frac{\tau_d s}{N}\right)} r(t) - K\left[1 + \frac{\tau_d s}{1+\frac{\tau_d s}{N}} + \frac{1}{\tau_i s\left(1+\frac{\tau_d s}{N}\right)}\right] y(t),$$

$$= \frac{T_c}{R_c} r(t) - \frac{S_c}{R_c} y(t) \qquad T_c = ? \quad S_c(0) = ?$$

6. r Not Sent through Derivative Mode

$$u(t) = \frac{K}{\tau_i s\left(1+\frac{\tau_d s}{N}\right)} r(t) - K\left[1 + \frac{\tau_d s}{1+\frac{\tau_d s}{N}} + \frac{1}{\tau_i s\left(1+\frac{\tau_d s}{N}\right)}\right] y(t)$$

Substituting $s \leftrightarrow (1-z^{-1})/T_s$ and simplifying

$$u(n) = \frac{T_c}{R_c} r(n) - \frac{S_c}{R_c} y(n)$$

$$S_c(z) = s_0 + s_1 z^{-1} + s_2 z^{-2}, \quad s_0 = K\left[1 - N r_1 - \frac{N r_1 T_s^2}{\tau_i \tau_d}\right]$$

$$s_1 = K[r_1(1+2N) - 1], \quad s_2 = -K r_1 (1+N)$$

$$T_c = -\frac{K N T_s^2 r_1}{\tau_i \tau_d}, \quad R_c = (1-z^{-1})(1+r_1 z^{-1})$$

$$S_c = ?$$

7. Other PID Structures

- Reference signal sent only through integrator!
- No reference signal through proportional, derivative modes!

$$u(t) = K\left[-y(t) + \frac{1}{\tau_i s}e(t) - \tau_d s y(t)\right]$$

- Part of reference sent through proportional mode

$$u(t) = K\left[br(t) - y(t) + \frac{1}{s\tau_i}(r(t) - y(t)) - \frac{s\tau_d}{1 + \frac{s\tau_d}{N}}y(t)\right]$$

- Check the conditions for tracking of step inputs

8. Two Degrees of Freedom Pole Placement Controller

Consider a plant with transfer function

$$G(z) = z^{-k}\frac{B(z)}{A(z)},$$

- $B(z)$ and $A(z)$ are coprime.
- Want output y to be related to setpoint r as,

$$Y_m(z) = \gamma z^{-k}\frac{B_r}{\phi_{cl}}R(z)$$

- ϕ_{cl} is ch. polynomial from desired region analysis
- γ - chosen so that at steady state $Y_m = R$:

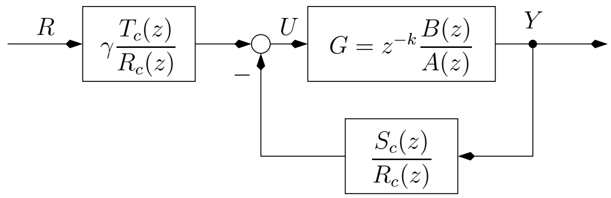
$$\gamma = \frac{\phi_{cl}(1)}{B_r(1)}$$

9. Two Degrees of Freedom Pole Placement Controller

Look for a controller of the form,

$$R_c(z)U(z) = \gamma T_c(z)R(z) - S_c(z)Y(z)$$

$R_c(z)$, $S_c(z)$ and $T_c(z)$ are polynomials in z^{-1}, to be determined.

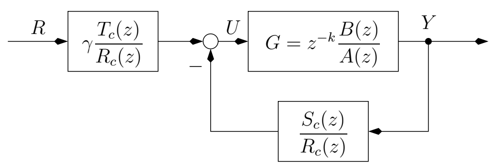


The controller has two components:

1. Feedback for internal stability.
2. Feedforward for Y to track R.

Known as the two degrees of freedom controller.

10. 2-DOF Pole Placement Controller - Basic Design



$$Y = \frac{z^{-k}\frac{B}{A}}{1 + z^{-k}\frac{B}{A}\frac{S_c}{R_c}}\gamma\frac{T_c}{R_c}R = \gamma z^{-k}\frac{BT_c}{AR_c + z^{-k}BS_c}R$$

Want Y to behave like Y_m:

$$Y_m(z) = \gamma z^{-k}\frac{B_r}{\phi_{cl}}R(z)$$

Equate the two and cancel the common factors:

$$\frac{BT_c}{AR_c + z^{-k}BS_c} = \frac{B_r}{\phi_{cl}}$$

$$\deg B_r < \deg B$$

11. 2-DOF Pole Placement Controller - Basic Design

$$\frac{BT_c}{AR_c + z^{-k}BS_c} = \frac{B_r}{\phi_{cl}}$$

$$\text{Let } B = B^g B^b, \quad A = A^g A^b$$

$$T_c = A^g T_1, \quad R_c = B^g R_1, \quad S_c = A^g S_1,$$

The last equation becomes

$$\frac{B^b B^g A^g T_1}{A^b A^g B^g R_1 + z^{-k} B^b B^g A^g S_1} = \frac{B_r}{\phi_{cl}}.$$

Cancelling good common factors,

$$\frac{B^b T_1}{A^b R_1 + z^{-k} B^b S_1} = \frac{B_r}{\phi_{cl}}$$

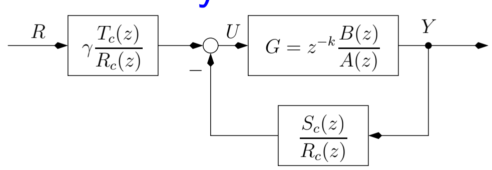
Equate numerator and denominator,

12. Aryabhatta Identify

$$B_r = B^b T_1$$

$$A^b R_1 + z^{-k} B^b S_1 = \phi_{cl}$$

Known as Aryabhatta Identity. It can be solved for R_1 and S_1

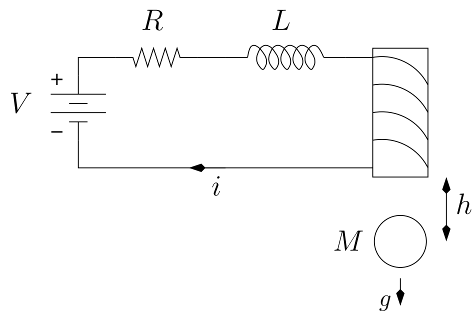


Desired closed loop relation:

$$Y_m(z) = \gamma z^{-k} \frac{B_r}{\phi_{cl}} R(z) = \gamma z^{-k} \frac{B^b T_1}{\phi_{cl}} R(z)$$

If $B^b \neq 1$, it will appear as a part of closed loop transfer function. i.e. bad zeros of original transfer function cannot be changed by feedback.

13. Magnetically Suspended Ball



- Current through coil induces magnetic force
- Magnetic force balances gravity
- Ball is suspended in midair - 1 cm from core
- Want to move to another equilibrium

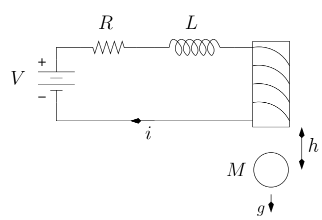
Force balance:

$$M\frac{d^2h}{dt^2} = Mg - \frac{Ki^2}{h}$$

Voltage balance

$$V = L\frac{di}{dt} + iR$$

14. Magnetically Suspended Ball - 2



Model equations:

$$M\frac{d^2h}{dt^2} = Mg - \frac{Ki^2}{h}$$
$$V = L\frac{di}{dt} + iR$$

In deviation variables:

$$0 = M\frac{d^2h_s}{dt^2} = Mg - \frac{Ki_s^2}{h_s}$$
$$M\frac{d^2\Delta h}{dt^2} = -K\left[\frac{i^2}{h} - \frac{i_s^2}{h_s}\right]$$

Linearize RHS:

$$\frac{i^2}{h} = \frac{i_s^2}{h_s} + 2\left.\frac{i}{h}\right|_{(i_s,h_s)}\Delta i - \left.\frac{i^2}{h^2}\right|_{(i_s,h_s)}\Delta h$$
$$= \frac{i_s^2}{h_s} + 2\frac{i_s}{h_s}\Delta i - \frac{i_s^2}{h_s^2}\Delta h$$

Substitute and simplify

$$M\frac{d^2\Delta h}{dt^2} = -K\left[\frac{i_s^2}{h_s} + 2\frac{i_s}{h_s}\Delta i - \frac{i_s^2}{h_s^2}\Delta h - \frac{i_s^2}{h_s}\right]$$
$$\frac{d^2\Delta h}{dt^2} = \frac{K}{M}\frac{i_s^2}{h_s^2}\Delta h - 2\frac{K}{M}\frac{i_s}{h_s}\Delta i.$$

Voltage balance in deviation:

$$\Delta V = L\frac{d\Delta i}{dt} + R\Delta i$$

15. Magnetically Suspended Ball - Continued 3

Force balance:

$$\frac{d^2\Delta h}{dt^2} = \frac{K}{M}\frac{i_s^2}{h_s^2}\Delta h - 2\frac{K}{M}\frac{i_s}{h_s}\Delta i$$

Voltage balance:

$$\Delta V = L\frac{d\Delta i}{dt} + R\Delta i$$

Define new variables

$$x_1 \triangleq \Delta h$$
$$x_2 \triangleq \Delta \dot{h}$$
$$x_3 \triangleq \Delta i$$
$$u \triangleq \Delta V$$

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{K}{M}\frac{i_s^2}{h_s^2}x_1 - 2\frac{K}{M}\frac{i_s}{h_s}x_3$$
$$\frac{dx_3}{dt} = -\frac{R}{L}x_3 + \frac{1}{L}u$$

In matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{K}{M}\frac{i_s^2}{h_s^2} & 0 & -2\frac{K}{M}\frac{i_s}{h_s} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u$$

This is of the form

$$\dot{x}(t) = Fx(t) + Gu(t)$$

16. Magnetically Suspended Ball - Continued 4

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{K}{M}\frac{i_s^2}{h_s^2} & 0 & -2\frac{K}{M}\frac{i_s}{h_s} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u$$

M	Mass of ball	0.05 Kg
L	Inductance	0.01 H
R	Resistance	1 Ω
K	Coefficient	0.0001
g	Acceleration due to gravity	9.81 m/s^2
h_s	Equilibrium Distance	0.01 m
i_s	Current at equilibrium	7A

17. Magnetically Suspended Ball - 5

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 981 & 0 & -2.801 \\ 0 & 0 & -100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} u$$

$$y = Cx,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$x_1 \triangleq \Delta h, \quad x_2 \triangleq \Delta \dot{h}$$

$$x_3 \triangleq \Delta i, \quad u \triangleq \Delta V$$

$$G(s) = \frac{-280.14}{s^3 + 100s^2 - 981s - 98100}$$

18. Magnetically Suspended Ball - Discrete Model

Sample continuous time model using $T_s = 0.01s$. Using myc2d.m,

$$G(z) = z^{-1}\frac{(-3.7209 \times 10^{-5} - 1.1873 \times 10^{-4}z^{-1} - 2.2597 \times 10^{-5}z^{-2})}{1 - 2.4668z^{-1} + 1.7721z^{-2} - 0.3679z^{-3}}$$

$$= z^{-1}\frac{-3.7209 \times 10^{-5}(1 + 2.9877z^{-1})(1 + 0.2033z^{-1})}{(1 - 1.3678z^{-1})(1 - 0.7311z^{-1})(1 - 0.3679z^{-1})} = z^{-k}\frac{B}{A}$$

Split A and B:

$$A = A^g A^b$$

$$A^g = (1 - 0.7311z^{-1})(1 - 0.3679z^{-1})$$

$$A^b = (1 - 1.3678z^{-1})$$

$$B^g = -3.7209 \times 10^{-5}(1 + 0.2033z^{-1})$$

$$B^b = (1 + 2.9877z^{-1})$$

19. Determination of ϕ_{cl}

$T_s = 0.01$. Transient specifications for a step input:

$$\begin{aligned}\text{Steady state error} &\le 2\%,\\ \text{Overshoot} = \varepsilon &\le 5\%\\ \text{Settling time} &\le 0.5s\end{aligned}$$

Guess

$$\begin{aligned}\text{Rise time} &\le 0.15s\\ N \le \text{rise time}/T_s &= 15\end{aligned}$$

Choose

$$\begin{aligned}N &= 15.\\ \omega &= \frac{\pi}{2N} = \frac{\pi}{30} = 0.1047\end{aligned}$$

20. Determination of ϕ_{cl}

Overshoot constraint:

$$\rho \le \varepsilon^{\omega/\pi} = 0.05^{0.1047/\pi} = 0.905$$

Choose

$$\rho = 0.905$$

Desired closed loop poles:

$$\begin{aligned}z &= \rho e^{\pm j\omega}\\ \phi_{cl}(z) &= 1 - 2z^{-1}\rho\cos\omega + \rho^2 z^{-2}\\ &= 1 - 1.8z^{-1} + 0.819z^{-2}\end{aligned}$$

21. Controller Design

$$A^g = (1 - 0.7311z^{-1})(1 - 0.3679z^{-1})$$
$$A^b = (1 - 1.3678z^{-1})$$
$$B^g = -3.7209 \times 10^{-5}(1 + 0.2033z^{-1})$$
$$B^b = (1 + 2.9877z^{-1})$$
$$\phi_{cl} = 1 - 1.8z^{-1} + 0.819z^{-2}$$

Solve

$$A^b R_1 + z^{-k} B^b S_1 = \phi_{cl}$$
$$(1 - 1.3678z^{-1})R_1 + z^{-1}(1 + 2.9877z^{-1})S_1 = 1 - 1.8z^{-1} + 0.819z^{-2}$$

Using xdync.m, we get

$$S_1 = 0.0523$$
$$R_1 = 1 - 0.4845z^{-1}.$$

22. ball_basic.m

```
% Magnetically suspended ball problem
% Operating conditions
M = 0.05; L = 0.01; R = 1; K = 0.0001; g = 9.81;
%
% Equilibrium conditions
hs = 0.01; is = sqrt(M*g*hs/K);
%
% State space matrices
a21 = K*is^2/M/hs^2; a23 = - 2*K*is/M/hs; a33 = - R/L;
b3 = 1/L;
a = [0 1 0; a21 0 a23; 0 0 a33];
b = [0; 0; b3]; c = [1 0 0]; d = 0;

% Transfer functions
G = ss(a,b,c,d); Ts = 0.01; [B,A,k] = myc2d(G,Ts);
[num,den] = tfdata(G,'v');

% Transient specifications
rise = 0.15; epsilon = 0.05;
phi = desired(Ts,rise,epsilon);
```

```
% Controller design
[Rc,Sc,Tc,gamma] = pp_basic(B,A,k,phi);

% Setting up simulation parameters for basic.mdl
st = 0.0001; % desired change in h, in m.
t_init = 0; % simulation start time
t_final = 0.5; % simulation end time

% Setting up simulation parameters for c_ss_cl
N_var = 0; xInitial = [0 0 0]; N = 1; C = 0; D = 1;
```

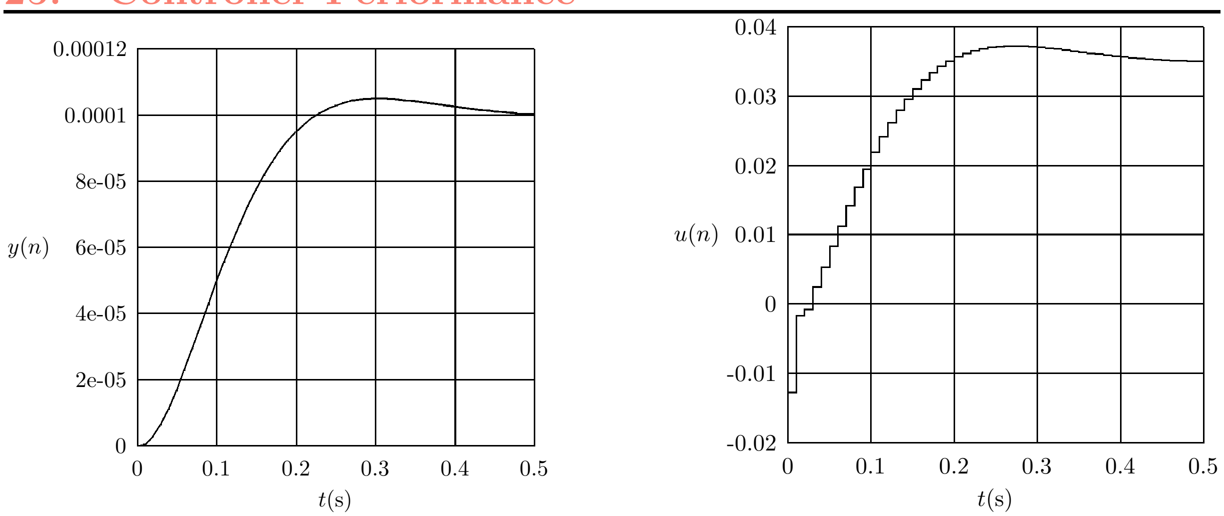

23. desired.m

```
% function [phi,dphi] = desired(Ts,rise,epsilon)
% Based on transient requirements,
% calculates closed loop characteristic polynomial
%
function [phi,dphi] = desired(Ts,rise,epsilon)

Nr = rise/Ts; omega = pi/2/Nr; rho = epsilon^(omega/pi);
phi = [1 -2*rho*cos(omega) rho^2]; dphi = length(phi)-1;
```

24. pp_basic.m

```
% function [Rc,Sc,Tc,gamma] = pp_basic(B,A,k,phi)
% calculates pole placement controller
%
function [Rc,Sc,Tc,gamma] = pp_basic(B,A,k,phi)

% Setting up and solving Aryabhatta identity
[Ag,Ab] = polsplit2(A); dAb = length(Ab) - 1;
[Bg,Bb] = polsplit2(B); dBb = length(Bb) - 1;
[zk,dzk] = zpowk(k);
[N,dN] = polmul(Bb,dBb,zk,dzk);
dphi = length(phi) - 1;
[S1,dS1,R1,dR1] = xdync(N,dN,Ab,dAb,phi,dphi);
%
% Determination of control law
Rc = conv(Bg,R1); Sc = conv(Ag,S1);
Tc = Ag; gamma = sum(phi)/sum(Bb);
```


25. Controller Performance



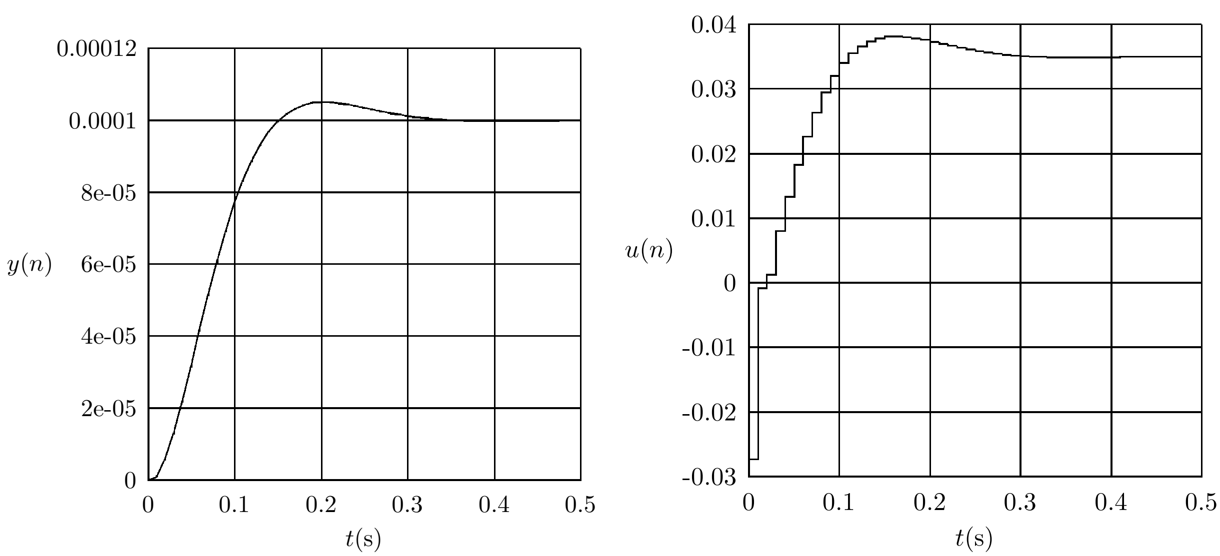
- Overshoot constraint, settling time constraints are met
- Settling time constraint also is met
- The auxiliary condition on rise time is not met, however
- Try reducing the required rise time

26. Controller Design with Rise Time = 0.1s

$$
\begin{aligned}
A^g &= (1 - 0.7311z^{-1})(1 - 0.3679z^{-1}) \\
A^b &= (1 - 1.3678z^{-1}) \\
B^g &= -3.7209 \times 10^{-5}(1 + 0.2033z^{-1}) \\
B^b &= (1 + 2.9877z^{-1}) \\
\varepsilon &= 0.05 \\
r &= 0.8609 \\
\omega &= 0.1571 \\
\phi_{cl} &= 1 - 1.7006z^{-1} + 0.7411z^{-2} \\
R_1 &= 1 - 0.3984z^{-1} \\
S_1 &= 0.0657
\end{aligned}
$$

$$(1 - 1.3678z^{-1})R_1 + z^{-1}(1 + 2.9877z^{-1})S_1 = 1 - 1.7006z^{-1} + 0.7411z^{-2}$$

27. Controller Design with Rise Time = 0.1s



All requirements have been met