1. Auto Covariance Function - Example

Find the ACF of $\{u(n)\} = \{1, 2\}$. Mean $= m_u = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} u(n)$

$$m_u = \frac{1}{2}\sum_{k=0}^{1} u(k) = \frac{1}{2}(u(0) + u(1)) = 1.5$$

ACF $= r_{uu}(l) = \lim_{N\to\infty} \frac{1}{2N}\sum_{k=-N}^{N}(u(k) - m_u)(u(k-l) - m_u)$

$$\begin{aligned}
r_{uu}(0) &= \sum_{k=0}^{1}(u(k) - 1.5)^2 = (-0.5)^2 + 0.5^2 = 0.5 \\
r_{uu}(1) &= \sum_{k=0}^{1}(u(k) - 1.5)(u(k-1) - 1.5) \\
&= (u(1) - 1.5)(u(0) - 1.5) = 0.5 \times (-0.5) = -0.25 \\
r_{uu}(-1) &= \sum_{k=0}^{1}(u(k) - 1.5)(u(k+1) - 1.5) \\
&= \frac{1}{2}(u(0) - 1.5)(u(1) - 1.5) = (-0.5) \times 0.5 = -0.25
\end{aligned}$$

2. Cross Covariance Function - Example

Estimate of CCF:

$$r_{uy}(l) = \lim_{N\to\infty} \frac{1}{2N}\sum_{k=-N}^{N}(u(k) - m_u)(y(k-l) - m_y)$$

To calculate $r_{uy}(l)$, $l > 0$, shift y by l points to right multiply, add. Check:

$$r_{uy}(l) = r_{yu}(-l)$$

Causality: Current output cannot be correlated with a future input.

$$r_{uy}(l) = r_{yu}(-l) = 0, \ \forall l < 0$$

3. White Noise

- The discrete-time white-noise sequence $\{\xi(k)\}$ is a set of independent, identically distributed (i.i.d.) values belonging to a stationary stochastic process.
- The mean of white noise is zero.
- The ACF of a white-noise sequence is given by:

$$\gamma_{\xi\xi}(k) = \sigma_\xi^2 \delta(k) = \begin{cases} \sigma_\xi^2 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- The Z-transform of ACF of white noise is σ_ξ^2.
- White noise has infinite energy $\Rightarrow$ its Fourier Transform does not exist.
- But the Fourier Transform of ACF (power spectrum) of white-noise is constant and given by,

$$\Phi_{\xi\xi}(\omega) = \sigma_\xi^2, \quad \forall \omega$$

4. Use of ACF: An Example

Determine a with white noise u:

$$y(n) - ay(n-1) = \xi(n), \quad (1)$$

Multiply Eq. 1 by $\xi(n-k)$ and sum:

$$\gamma_{y\xi}(k) - a\gamma_{y\xi}(k-1) = \gamma_{\xi\xi}(k). \quad (2)$$

Because the system is causal,

$$\gamma_{y\xi}(n) = 0, \ \forall n < 0.$$

By evaluating Eq. 2 for $k = 0$ and 1,

$$\begin{aligned} \gamma_{y\xi}(0) &= \gamma_{\xi\xi}(0) = \sigma^2, \\ \gamma_{y\xi}(1) &= a\,\sigma^2. \end{aligned}$$

Multiply Eq. 1 by $y(n-k)$ and sum:

$$\gamma_{yy}(k) - a\,\gamma_{yy}(k-1) = \gamma_{y\xi}(-k)$$

Evaluate for $k = 0, 1$:

$$\begin{aligned} \gamma_{yy}(0) - a\gamma_{yy}(1) &= \sigma^2 \\ \gamma_{yy}(1) - a\gamma_{yy}(0) &= 0 \end{aligned}$$

Solving,

$$a = \frac{\gamma_{yy}(1)}{\gamma_{yy}(0)}$$

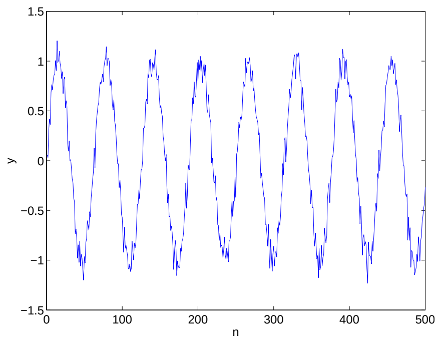
$$\gamma_{yy}(0) = \frac{\sigma^2}{1-a^2}$$

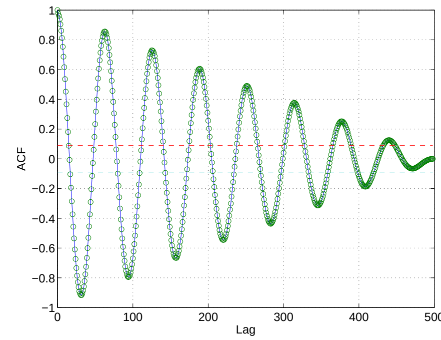
From top equation,

$$\begin{aligned} \gamma_{yy}(k) &= a\gamma_{yy}(k-1) = a^2\gamma_{yy}(k-2) \\ &= \cdots = \gamma_{yy}(0)a^k \end{aligned}$$

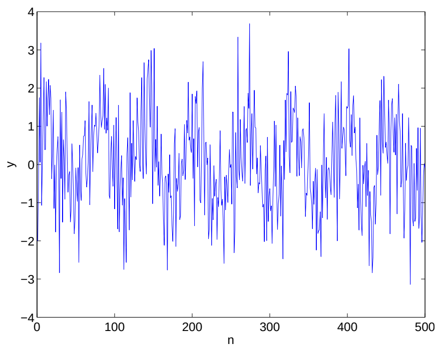
5. ACF of Noisy Periodic Signal is Periodic

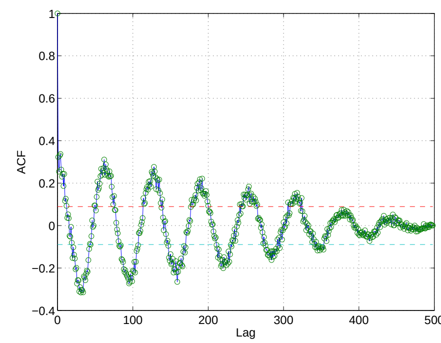
$\sin 0.1n + 0.1\xi(n)$:





$\sin 0.1n + \xi(n)$:





6. ARMA, AR Processes

ARMA: Auto Regressive Moving Average. Required to model noise process.

$$y(n) + a_1 y(n-1) + \cdots + a_p y(n-p) = \xi(n) + c_1 \xi(n-1) + \cdots + c_q \xi(n-q)$$

If $q = 0$, we obtain an AR(p) process:

$$y(n) + a_1 y(n-1) + \cdots + a_p y(n-p) = \xi(n) = \left(1 + \sum_{k=1}^{p} a_k z^{-k}\right) y(n)$$

or, equivalently,

$$y(n) = \frac{1}{\left(1 + \sum_{k=1}^{p} a_k z^{-k}\right)} \xi(n) = \frac{1}{A(z)} \xi(n)$$

- A random sequence whose value $y(n)$ can be represented as a weighted finite aggregate of the p previous values plus a white-noise sequence $\xi(n)$ is said to be an AR process of order p.

7. ARMA, MA Processes

Recall ARMA process:

$$y(n) + a_1 y(n-1) + \cdots + a_p y(n-p) = \xi(n) + c_1 \xi(n-1) + \cdots + c_q \xi(n-q)$$

When $p = 0$, arrive at MA(q) process:

$$\begin{aligned} y(n) &= \xi(n) + c_1 \xi(n-1) + \cdots + c_q \xi(n-q) \\ &= \left(1 + \sum_{k=1}^{q} c_k z^{-k}\right) \xi(n) = C(z)\xi(n) \end{aligned}$$

- A random sequence whose value $y(n)$ can be represented as a finite combination of the past white-noise sequence e plus a random error $\xi(n)$ is said to be an MA process of order q.

ARMA contains both AR and MA components. Using the above procedure,

$$y(n) = \frac{C(z)}{A(z)}\xi(n) = \frac{1 + \sum_{n=1}^{q} c_n z^{-n}}{1 + \sum_{n=1}^{p} a_n z^{-n}}\xi(n)$$

8. Procedure to Distinguish MA(1) Processes

Develop a method to determine the order of the MA(1) process:

$$y(k) = \xi(k) + c_1 \xi(k-1).$$

We will begin with the calculation of ACF at zero lag:

$$\gamma_{yy}(0) = \mathscr{E}(y(k), y(k)) = \mathscr{E}\left[(\xi(k) + c_1\xi(k-1))(\xi(k) + c_1\xi(k-1))\right]$$

Because $\xi(k)$ is white, expectation of cross products are zero. We obtain

$$\gamma_{yy}(1) = (1 + c_1^2)\sigma_\xi^2$$

Next we determine ACF at lag 1:

$$\gamma_{yy}(1) = \mathscr{E}(y(k), y(k-1)) = \mathscr{E}\left[(\xi(k) + c_1\xi(k-1))(\xi(k-1) + c_1\xi(k-2))\right]$$

Once again invoking the fact that e is white and cancelling the cross terms, we obtain

$$\gamma_{yy}(1) = \mathscr{E}(c_1 \xi^2(k-1)) = c_1 \sigma_\xi^2$$

For all other lags, ACF is zero. That is,

$$\gamma_{yy}(l) = 0, \quad l > 1$$

9. Procedure to Distinguish MA(q) Processes

Start with a general MA(q) process:

$$y(n) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q)$$

Multiplying by $y(n)$ and taking expectation

$$\gamma_{yy}(0) = \gamma_{y\xi}(0) + c_1\gamma_{y\xi}(1) + \cdots + c_q\gamma_{y\xi}(q)$$

Multiplying by $y(n-1)$ and taking expectation,

$$\gamma_{yy}(1) = c_1\gamma_{y\xi}(0) + c_2\gamma_{y\xi}(1) + \cdots + c_q\gamma_{y\xi}(q-1)$$

noting

$$\mathscr{E}[y(n-1)\xi(n)] = 0$$

from causality principle: for causal systems, the output cannot depend on future input $\xi(n)$. Continuing the above process and stacking the resulting equations, we arrive at

$$\begin{bmatrix}\gamma_{yy}(0)\\ \gamma_{yy}(1)\\ \vdots\\ \gamma_{yy}(q)\end{bmatrix} = \begin{bmatrix}1 & c_1 & \cdots & c_{q-1} & c_q\\ c_1 & c_2 & \cdots & c_q & 0\\ \vdots & & & & \vdots\\ c_q & 0 & \cdots & & 0\end{bmatrix}\begin{bmatrix}\gamma_{y\xi}(0)\\ \gamma_{y\xi}(1)\\ \vdots\\ \gamma_{y\xi}(q)\end{bmatrix}$$

All the terms below the diagonal are zero. It is clear that

$$\gamma_{yy}(k) = 0, \ \forall k > q$$

Thus we obtain the rule that for MA(q) process, a plot of $\{\gamma_{yy}(k)\}$ *vs.* k becomes zero for all $k > q$.

10. Example: MA(2) Process

Calculate $\{\gamma_{yy}(k)\}$ for

$$y(n) = \xi(n) + \xi(n-1) - 0.5\xi(n-2) \quad (3)$$

Multiplying by $y(n-k)$, $k \geq 0$

$$\begin{aligned}\gamma_{yy}(0) &= \gamma_{y\xi}(0) + \gamma_{y\xi}(1) - 0.5\gamma_{y\xi}(2)\\ \gamma_{yy}(1) &= \gamma_{y\xi}(0) - 0.5\gamma_{y\xi}(1)\\ \gamma_{yy}(2) &= -0.5\gamma_{y\xi}(0)\\ \gamma_{yy}(k) &= 0, \quad k \geq 0\end{aligned} \quad (4)$$

Multiply 3 by $\xi(n)$, $\xi(n-1)$ and $\xi(n-2)$ and take expectation:

$$\gamma_{y\xi}(0) = \gamma_{\xi\xi}(0) = \sigma_\xi^2$$

because $\{\xi(n)\}$ is white.

$$\begin{aligned}\gamma_{y\xi}(1) &= \gamma_{\xi\xi}(0) = \sigma_\xi^2\\ \gamma_{y\xi}(2) &= -0.5\gamma_{\xi\xi}(0) = -0.5\sigma_\xi^2\end{aligned}$$

Substituting these in Eq. 4,

$$\begin{aligned}\gamma_{yy}(0) &= (1 + 1 + 0.25)\sigma_\xi^2\\ &= 2.25\sigma_\xi^2\\ \gamma_{yy}(1) &= (1 - 0.5)\sigma_\xi^2\\ &= 0.5\sigma_\xi^2\\ \gamma_{yy}(2) &= -0.5\sigma_\xi^2\\ \gamma_{yy}(k) &= 0, \ k \geq 3,\end{aligned}$$

as expected. Known as theoretical prediction approach.

11. Matlab Code to Calculate ACF

```
% Define the model
m = idpoly(1,[],[1,1,-0.5]);

% Generate noise and the response
e = 0.1*randn(100000,1);
y = sim(m,e); z = [y e];

% Plot noise and plant output
subplot(2,1,1), plot(y(1:500))
title('Plant output, noise input vs. time',...
      'FontSize',14)
ylabel('Plant output y','FontSize',14)
subplot(2,1,2), plot(e(1:500))
ylabel('Noise input e','FontSize',14)
xlabel('Sampling instant, k','FontSize',14)

% Calculate covariance and plot it
ryy = xcov(y,'coeff');
figure, plotacf(y,1,11,1);
```


12. Input-Output Plots and ACF