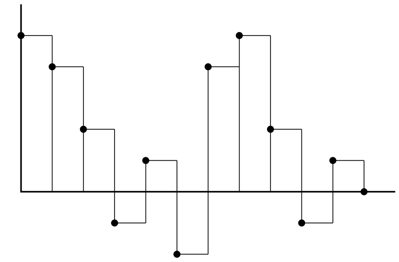
1. s to Z-Domain Transfer Function



1. Determine the step response of the continuous transfer function $y_s(t)$.
2. Discretize the step response to arrive at $y_s(nT_s)$.

1. Z-transform the step response to obtain $Y_s(z)$.
2. Divide the function obtained in the above step by the Z-transform of a step input, namely, $z/(z-1)$.

- $G_a(s)$: Laplace transfer function
- $G(z)$: Z-transfer function

$$G(z) = \frac{z-1}{z} Z\left[\mathcal{L}^{-1}\frac{G_a(s)}{s}\right]$$

Step Response Equivalence = ZOH Equivalence

2. Important Result from Differentiation

Problem 4.9 in Text: Consider

$$1(n)a^n \leftrightarrow \frac{z}{z-a} = \sum_{n=0}^{\infty} a^n z^{-n},$$

Differentiating w.r.t. a,

$$\frac{z}{(z-a)^2} = \sum_{n=0}^{\infty} n a^{n-1} z^{-n}$$

$$n a^{n-1} 1(n) \leftrightarrow \frac{z}{(z-a)^2}$$

$$n(n-1)a^{n-2}1(n) \leftrightarrow \frac{2z}{(z-a)^3}$$

Example

$$n^2 1(n) = [n(n-1)+n]\, 1(n)$$

- Notice that $a = 1$ now.
- Take Z-transform,

$$\leftrightarrow \frac{2z}{(z-1)^3} + \frac{z}{(z-1)^2}$$
$$= \frac{z^2+z}{(z-1)^3}$$

3. ZOH Equivalence of $1/s$

The step response of $1/s$ is $1/s^2$. In time domain, it is given by,

$$y_s(t) = \mathcal{L}^{-1}\frac{1}{s^2} = t$$

Sampling it with a period of T_s,

$$y_s(nT_s) = nT_s$$

Taking Z-transforms

$$Y_s(z) = \frac{T_s z}{(z-1)^2}$$

Divide by $z/(z-1)$, to get the ZOH equivalent discrete domain transfer function

$$G(z) = \frac{T_s}{z-1}$$

4. ZOH Equivalence of $1/s^2$

The step response of $1/s^2$ is $1/s^3$. In time domain, it is given by,

$$y_s(t) = \mathcal{L}^{-1}\frac{1}{s^3} = \frac{1}{2}t^2.$$

Sampling it with a period of T_s,

$$y_s(nT_s) = \frac{1}{2}n^2T_s^2$$

Take Z-transform

$$Y_s(z) = \frac{T_s^2 z(z+1)}{2(z-1)^3}$$

Dividing by $z/(z-1)$, we get

$$G(z) = \frac{T_s^2(z+1)}{2(z-1)^2}$$

5. ZOH Equivalent First Order Transfer Function

Find the ZOH equivalent of $K/(\tau s+1)$.

$$Y_s(s) = \frac{1}{s}\frac{K}{\tau s+1} = K\left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right]$$
$$y_s(t) = K\left[1 - e^{-t/\tau}\right], \quad t > 0$$
$$y_s(nT_s) = K\left[1 - e^{-nT_s/\tau}\right]1(n) = K\left[1(n) - e^{-nT_s/\tau}1(n)\right]$$
$$Y_s(z) = K\left[\frac{z}{z-1} - \frac{z}{z-e^{-T_s/\tau}}\right] = \frac{Kz(1-e^{-T_s/\tau})}{(z-1)(z-e^{-T_s/\tau})}$$

Dividing by $z/(z-1)$, we get

$$G(z) = \frac{K(1-e^{-T_s/\tau})}{z-e^{-T_s/\tau}}$$

6. ZOH Equivalent First Order Transfer Function - Example

Sample at $T_s = 0.5$ and find ZOH equivalent trans. function of

$$G_a(s) = \frac{10}{5s+1}$$

Matlab Code:

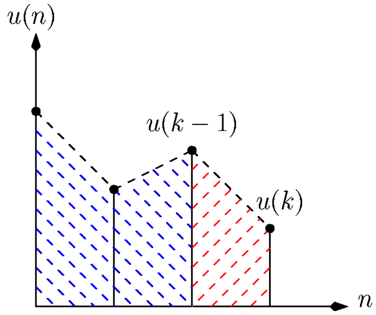
```
Ga = tf(10,[5 1]);
G = c2d(Ga,0.5);
```

Matlab output is,

$$G(z) = \frac{0.9546}{z-0.9048} = \frac{10(1-e^{-0.1})}{z-e^{-0.1}}$$

In agreement with the formula in the previous slide

7. Discrete Integration



$$y(k) = \text{blue shaded area} + \text{red shaded area}$$

$$y(k) = y(k-1) + \text{red shaded area}$$

$$y(k) = y(k-1) + \frac{T_s}{2}\left[u(k) + u(k-1)\right]$$

Take Z-transform:

$$Y(z) = z^{-1}Y(z) + \frac{T_s}{2}\left[U(z) + z^{-1}U(z)\right]$$

Bring all Y to left side:

$$Y(z) - z^{-1}Y(z) = \frac{T_s}{2}\left[U(z) + z^{-1}U(z)\right]$$

$$(1 - z^{-1})Y(z) = \frac{T_s}{2}(1 + z^{-1})U(z)$$

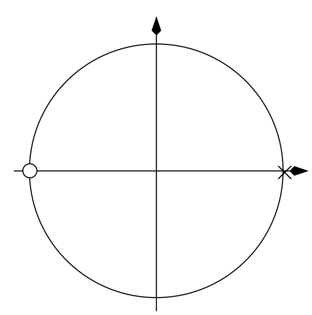
8. Transfer Function for Discrete Integration

Recall from previous slide

$$(1 - z^{-1})Y(z) = \frac{T_s}{2}(1 + z^{-1})U(z)$$

$$Y(z) = \frac{T_s}{2}\frac{1 + z^{-1}}{1 - z^{-1}}U(z)$$

$$= \frac{T_s}{2}\frac{z+1}{z-1}U(z)$$



$$\frac{1}{s} \leftrightarrow \frac{T_s}{2}\frac{z+1}{z-1}$$

Integrator has a transfer function,

$$G_I(z) = \frac{T_s}{2}\frac{z+1}{z-1}$$

A low pass filter!

9. Derivative Mode

- Integral Mode: $\frac{1}{s} \leftrightarrow \frac{T_s}{2}\frac{z+1}{z-1}$
- Derivative Mode: $s \leftrightarrow \frac{2}{T_s}\frac{z-1}{z+1}$
- High pass filter
- Has a pole at $z = -1$. Hence produces in partial fraction expansion, a term of the form

$$\frac{z}{z+1} \leftrightarrow (-1)^n$$

- Results in wildly oscillating control effort.

10. Derivative Mode - Other Approximations

Backward difference:

$$\begin{aligned} y(k) &= y(k-1) + T_s u(k) \\ (1-z^{-1})Y(z) &= T_s U(z) \\ Y(z) &= T_s \frac{1}{1-z^{-1}} = T_s \frac{z}{z-1} U(z) \\ \frac{1}{s} &\leftrightarrow T_s \frac{z}{z-1} \end{aligned}$$

Forward difference:

$$\begin{aligned} y(k) &= y(k-1) + T_s u(k-1) \\ (1-z^{-1})Y(z) &= T_s z^{-1} U(z) \\ Y(z) &= T_s \frac{z^{-1}}{1-z^{-1}} U(z) = \frac{T_s}{z-1} U(z) \\ \frac{1}{s} &\leftrightarrow \frac{T_s}{z-1} \end{aligned}$$

Both derivative modes are high pass, no oscillations, same gains

11. PID Controller

Proportional Mode: Most popular control mode. Increase in proportional mode generally results in

- Decreased steady state offset and increased oscillations

Integral Mode: Used to remove steady state offset. Increase in integral mode generally results in

- Zero steady state offset
- Increased oscillations

Derivative Mode: Mainly used for prediction purposes. Increase in derivative mode generally results in

- Decreased oscillations and improved stability
- Sensitive to noise

The most popular controller in industry.

12. PID Controller - Basic Design

Let the input to the controller by $E(z)$ and the output from it be $U(z)$. If gain is K, τ_i is integral time and τ_d is derivative time,

$$u(t) = K\left[e(t) + \frac{1}{\tau_i}\int_0^t e(t)dt + \tau_d\frac{de(t)}{dt}\right]$$

$$U(s) = K(1 + \frac{1}{\tau_i s} + \tau_d s)E(s)$$

$$U(s) \triangleq \frac{S_c(s)}{R_c(s)}E(s)$$

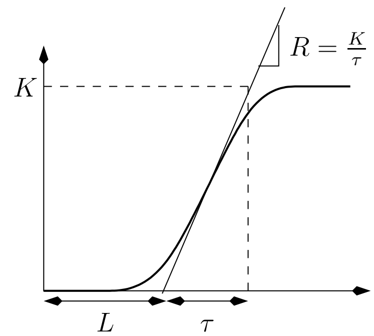
If integral mode is present, $R_c(0) = 0$. Filtered derivative mode:

$$u(t) = K\left(1 + \frac{1}{\tau_i s} + \frac{\tau_d s}{1 + \frac{\tau_d s}{N}}\right)e(t)$$

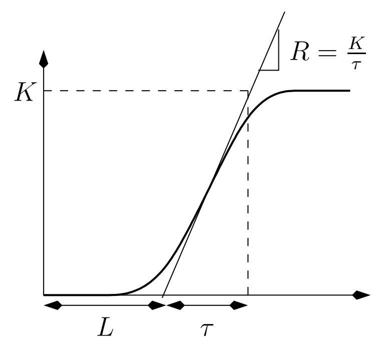
N is a large number, of the order of 100.

13. Reaction Curve Method - Ziegler Nichols Tuning

- Applicable only to stable systems
- Give a unit step input to a stable system and get
  1. the time lag after which the system starts responding (L),
  2. the steady state gain (K) and
  3. the time the output takes to reach the steady state, after it starts responding (τ)



14. Reaction Curve Method - Ziegler Nichols Tuning



- Let the slope of the response be calculated as $R = \dfrac{K}{\tau}$. Then the PID settings are given below:

	K_p	τ_i	τ_d
P	1/RL		
PI	0.9/RL	3L	
PID	1.2/RL	2L	0.5L

Consistent units should be used

15. Stability Method - Ziegler Nichols Tuning

Another way of finding the PID tuning parameters is as follows.

- Close the loop with a proportional controller
- Gain of controller is increased until the closed loop system becomes unstable
- At the verge of instability, note down the gain of the controller (K_u) and the period of oscillation (P_u)
- PID settings are given below:

	K_p	τ_i	τ_d
P	$0.5K_u$		
PI	$0.45K_u$	$P_u/1.2$	
PID	$0.6K_u$	$P_u/2$	$P_u/8$

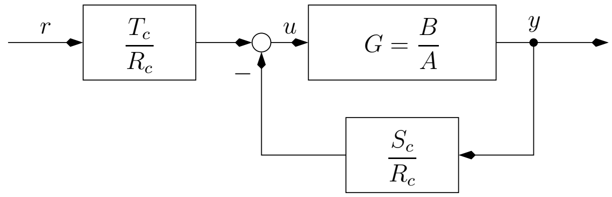
Consistent units should be used

16. Design Procedure

A common procedure to design discrete PID controller:

- Tune continuous PID controller by any popular technique
- Get continuous PID settings
- Discretize using the method discussed now or the ZOH equivalent method discussed earlier
- Direct digital design techniques

17. 2-DOF Controller



$$u = \frac{T_c}{R_c} r - \frac{S_c}{R_c} y$$

It is easy to arrive at the following relation between r and y.

$$y = \frac{T_c}{R_c} \frac{B/A}{1 + BS_c/AR_c} r = \frac{BT_c}{AR_c + BS_c} r$$

Error transfer function:

$$e = r - y = \left(1 - \frac{BT_c}{AR_c + BS_c}\right) r = \frac{AR_c + BS_c - BT_c}{AR_c + BS_c} r$$

18. Offset-Free Tracking of Steps with Integral Action

$$E(z) = \frac{A(z)R_c(z) + B(z)S_c(z) - B(z)T_c(z)}{A(z)R_c(z) + B(z)S_c(z)} R(z)$$

$$\lim_{n\to\infty} e(n) = \lim_{z\to 1} \frac{z-1}{z} \frac{A(z)R_c(z) + B(z)S_c(z) - B(z)T_c(z)}{A(z)R_c(z) + B(z)S_c(z)} \frac{z}{z-1}$$

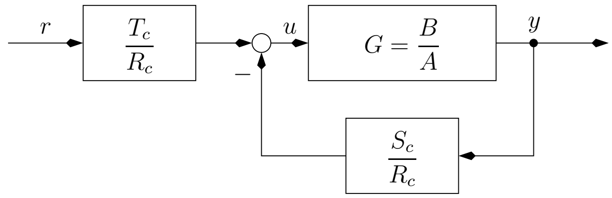
Because the controller has an integral action, $R_c(1) = 0$:

$$e(\infty) = \left.\frac{S_c(z) - T_c(z)}{S_c(z)}\right|_{z=1} = \frac{S_c(1) - T_c(1)}{S_c(1)}$$

This condition can be satisfied if one of the following is met:

$$\begin{aligned} T_c &= S_c \\ T_c &= S_c(1) \\ T_c(1) &= S_c(1) \end{aligned}$$

19. $T_c = S_c$: Offset-Free Tracking with Integral Action



$T_c = S_c$ results in

$$U(s) = K\left(1 + \frac{1}{\tau_i s} + \tau_d s\right)E(s)$$

Integral Mode: $\frac{1}{s} \leftrightarrow \frac{T_s}{2}\frac{z+1}{z-1}$. Derivative Mode: $s \leftrightarrow \frac{1}{T_s}\frac{z-1}{z}$:

$$U(z) = K\left[1 + \frac{1}{\tau_i}\frac{T_s}{2}\frac{z+1}{z-1} + \frac{\tau_d}{T_s}\frac{z-1}{z}\right]E(z)$$

20. $T_c = S_c$: Offset-Free Tracking with Integral Action

$$U(z) = K\left[1 + \frac{1}{\tau_i}\frac{T_s}{2}\frac{z+1}{z-1} + \frac{\tau_d}{T_s}\frac{z-1}{z}\right]E(z)$$

Simplifying this, obtain

$$\begin{aligned}
u(n+1) &= u(n) + s_0 e(n+1) + s_1 e(n) + s_2 e(n-1) \\
s_0 &= K\left[1 + \frac{T_s}{2\tau_i} + \frac{\tau_d}{T_s}\right] \\
s_1 &= K\left[-1 + \frac{T_s}{2\tau_i} - 2\frac{\tau_d}{T_s}\right] \\
s_2 &= K\frac{\tau_d}{T_s}
\end{aligned}$$

Smooth transfer from manual to auto mode. Bumpless transfer.