1. Fourier Transform of Discrete Time Aperiodic Signals

$$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} = X(e^{j\omega})$$

- $X(e^{j\omega})$ is periodic with period 2π. So, it has a Fourier Series.
- $x(n)$ can be calculated by integrating both sides

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega m}d\omega = \int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right]e^{j\omega m}d\omega$$

If the infinite sum converges,

$$= \sum_{n=-\infty}^{\infty} x(n)\int_{-\pi}^{\pi} e^{j\omega(m-n)}d\omega$$

$$= 2\pi\, x(m) + \sum_{n=-\infty, n\neq m}^{\infty} x(n)\int_{-\pi}^{\pi} e^{j\omega(m-n)}d\omega$$

$$= 2\pi\, x(m) + \sum_{n=-\infty, n\neq m}^{\infty} x(n)\left.\frac{e^{j\omega(m-n)}}{j(m-n)}\right|_{-\pi}^{\pi}$$

$$= 2\pi x(m)$$

From the above, solving for $x(m)$,

$$x(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega m}d\omega = \int_{-1/2}^{1/2} X(f)e^{j2\pi fm}df$$

2. Fourier Transform of a Moving Average Filter - Example

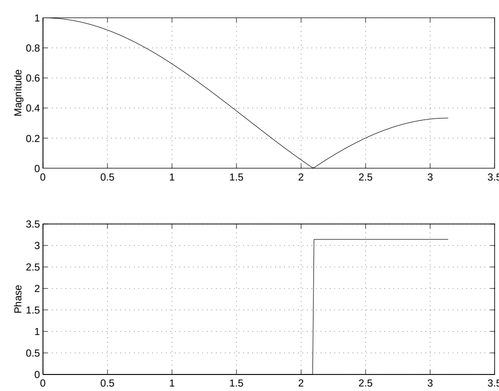
$$y(n) = \frac{1}{3}[u(n+1) + u(n) + u(n-1)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} g(k)u(n-k) = g(-1)u(n+1) + g(0)u(n) + g(1)u(n-1)$$

$$g(-1) = g(0) = g(1) = \frac{1}{3}$$

$$G\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} g(n)z^{-n}|_{z=e^{jw}} = \frac{1}{3}\left(e^{jw} + 1 + e^{-jw}\right) = \frac{1}{3}(1 + 2\cos w)$$

$$|G\left(e^{jw}\right)| = |\frac{1}{3}(1 + 2\cos w)|$$



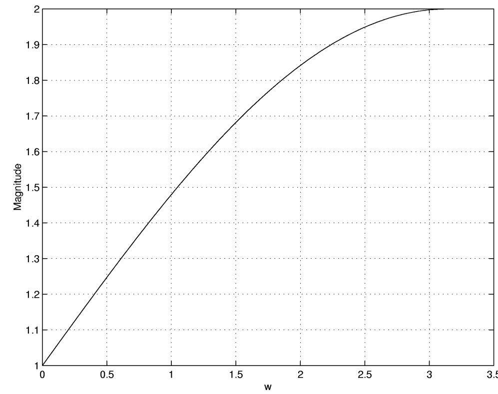
$$Arg(G) = \begin{cases} 0 & 0 \le w < \frac{2\pi}{3} \\ \pi & \frac{2\pi}{3} \le w < \pi \end{cases}$$

```
w = 0:0.01:pi;
subplot(2,1,1)
plot(w,abs(1+2*cos(w))/3),
grid, ylabel('Magnitude')
subplot(2,1,2)
plot(w,angle(1+2*cos(w))),
grid, xlabel('w'), ylabel('Phase')
```

3. Fourier Transform of a Differencing Filter - Example

$$
\begin{aligned}
y(n) &= u(n) - u(n-1) \\
g(0) &= 1, \quad g(1) = -1 \\
G\left(e^{jw}\right) &= G(z)|_{z=e^{jw}} \\
&= \sum_{n=-\infty}^{\infty} g(n) z^{-n}|_{z=e^{jw}} \\
&= 1 - e^{-jw} \\
&= e^{-jw/2}\left(e^{jw/2} - e^{-jw/2}\right) \\
&= 2je^{-jw/2} \sin\frac{w}{2} \\
|G| &= 2|\sin\frac{w}{2}|
\end{aligned}
$$

```
w = 0:0.01:pi;
plot(w,abs(1+sin(w/2))), grid, ...
xlabel('w'), ylabel('Magnitude')
```




```
sysd = tf([1 -1],1,-1);
w = logspace(-2,0.5);
bode(sysd,w)
```


4. Additional Properties of Fourier Transform

Symmetry of real and imaginary parts for real valued sequences

$$
\begin{aligned}
G\left(e^{jw}\right) &= \sum_{n=-\infty}^{\infty} g(n)e^{-jwn} \\
&= \sum_{n=-\infty}^{\infty} g(n)\cos wn - j\sum_{n=-\infty}^{\infty} g(n)\sin wn \\
G\left(e^{-jw}\right) &= \sum_{n=-\infty}^{\infty} g(n)e^{jwn} \\
&= \sum_{n=-\infty}^{\infty} g(n)\cos wn + j\sum_{n=-\infty}^{\infty} g(n)\sin wn
\end{aligned}
$$

Comparing the above two equations, we get

$$
\begin{aligned}
Re\left[G\left(e^{jw}\right)\right] &= Re\left[G\left(e^{-jw}\right)\right] \\
Im\left[G\left(e^{jw}\right)\right] &= -Im\left[G\left(e^{-jw}\right)\right]
\end{aligned}
$$

We can summarize these properties as

$$G\left(e^{jw}\right) = G^*\left(e^{-jw}\right)$$

Symmetry of magnitude and phase angle for real valued sequences

$$
\begin{aligned}
|G\left(e^{jw}\right)| &= \left[G\left(e^{jw}\right)G^*\left(e^{jw}\right)\right]^{1/2} \\
&= \left[G^*\left(e^{-jw}\right)G\left(e^{-jw}\right)\right]^{1/2} \\
&= |G(e^{-jw})|
\end{aligned}
$$

This shows that the magnitude is an even function. In a similar way,

$$Arg\left[G\left(e^{-jw}\right)\right] = -Arg\left[G\left(e^{jw}\right)\right]$$

$\Rightarrow$ Bode plots have to be drawn for w in $[0,\pi]$ only.

5. Sampling and Reconstruction

$$u(n) = u_a(nT_s), \quad -\infty < n < \infty \quad (1)$$

FT pair for analog, discrete signals:

$$U_a(F) = \int_{-\infty}^{\infty} u_a(t)e^{-j2\pi Ft}dt$$

$$u_a(t) = \int_{-\infty}^{\infty} U_a(F)e^{j2\pi Ft}dF. \quad (2)$$

$$U(f) = \sum_{n=-\infty}^{\infty} u(n)e^{-j2\pi fn}$$

$$u(n) = \int_{-1/2}^{1/2} U(f)e^{j2\pi fn}df \quad (3)$$

Substituting Eq. 3 and Eq. 2 in Eq. 1,

$$\int_{-1/2}^{1/2} U(f)e^{j2\pi fn}df = \int_{-\infty}^{\infty} U_a(F)e^{j2\pi FnT_s}dF$$

$$= \int_{-\infty}^{\infty} U_a(F)e^{j2\pi nF/F_s}dF$$

$$LHS = \int_{-1/2}^{1/2} U(f)e^{j2\pi fn}df$$

$$= \frac{1}{F_s}\int_{-F_s/2}^{F_s/2} U\left(\frac{F}{F_s}\right)e^{j2\pi nF/F_s}dF$$

$$RHS = \int_{-\infty}^{\infty} U_a(F)e^{j2\pi nF/F_s}dF$$

$$= \sum_{k=-\infty}^{\infty}\int_{(k-1/2)F_s}^{(k+1/2)F_s} U_a(F)e^{j2\pi nF/F_s}dF$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_s/2}^{F_s/2} U_a(Q+kF_s)e^{j2\pi n(Q+kF_s)/F_s}dQ$$

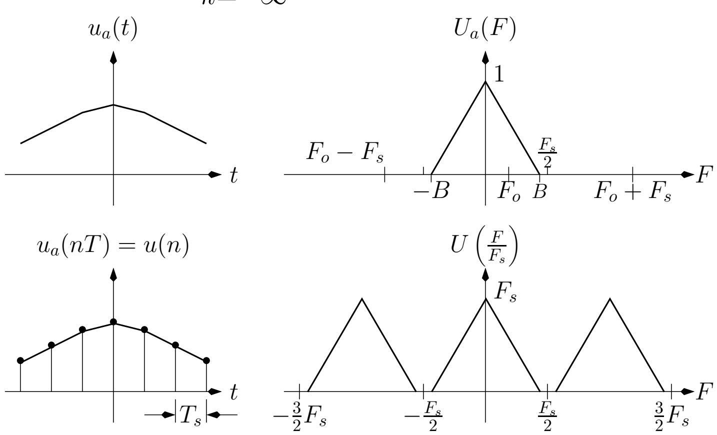
$$= \sum_{k=-\infty}^{\infty}\int_{-F_s/2}^{F_s/2} U_a(Q+kF_s)e^{j2\pi nQ/F_s}dQ$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_s/2}^{F_s/2} U_a(F+kF_s)e^{j2\pi nF/F_s}dF$$

$$= \int_{-F_s/2}^{F_s/2}\left(\sum_{k=-\infty}^{\infty} U_a(F+kF_s)\right)e^{j2\pi nF/F_s}dF$$

6. Fast Sampling Preserves All Required Information

$$U\left(\frac{F}{F_s}\right) = F_s\sum_{k=-\infty}^{\infty} U_a(F+kF_s) = F_s[\cdots + U_a(F-F_s) + U_a(F) + U_a(F+F_s) +$$



$$U\left(\frac{F_0}{F_s}\right) = F_sU_a(F_0)$$

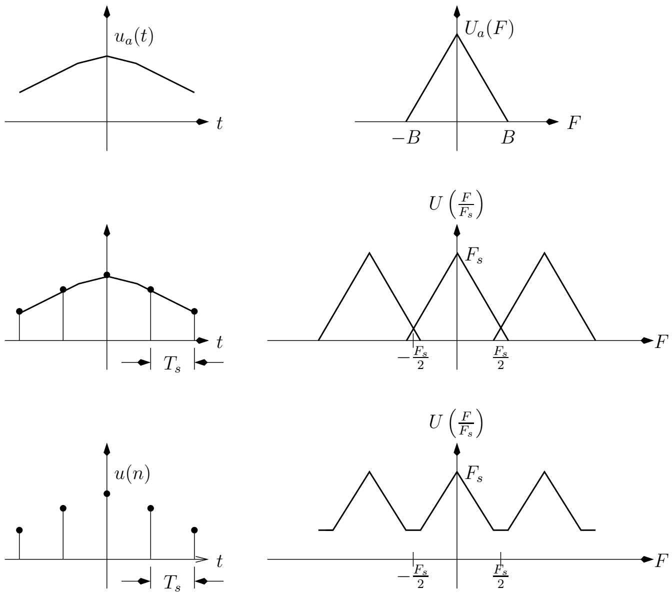
- U is scaled version of U_a - shape not affected by sampling. Can recover U_a from U.
- $U(F/F_s)$ is periodic in F with a period F_s:

$$U\left(\frac{F_0}{F_s}\right) = U\left(\frac{F_0+F_s}{F_s}\right) = U\left(\frac{F_0-F_s}{F_s}\right) = \cdots = U\left(\frac{F_0+kF_s}{F_s}\right),$$

$$k = \pm 1,\ \pm 2$$

7. Slow Sampling Results in Aliasing

$$U\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} U_a(F + kF_s) = F_s[\cdots + U_a(F_0 - F_s) + U_a(F_0) + U_a(F_0 +$$

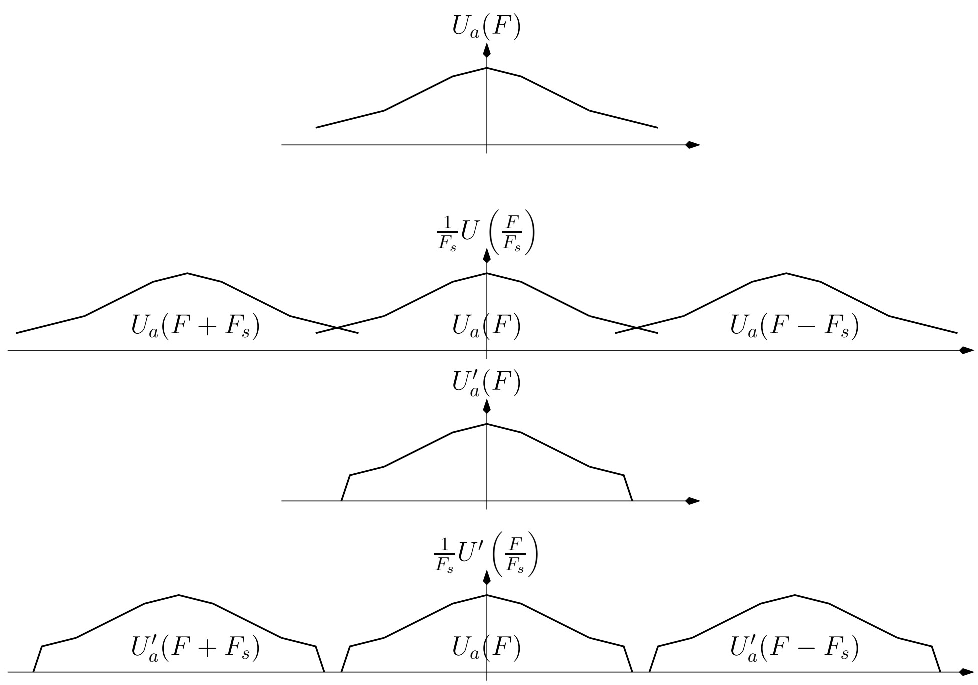


- Consequence of aliasing:

$$U\left(\frac{F_0}{F_s}\right)$$
$$= F_s[U_a(F_0 - F_s) + U_a(F_0)]$$
$$= F_s[\text{overlapping value} + U_a(F_0)]$$

- Cannot recover U_a from U, as the high frequency components have changed.

8. What to Do When Aliasing Cannot be Avoided?



9. Sampling Theorem

- Suppose highest frequency contained in an analog signal $u_a(t)$ is $F_{\max} = B$.
- It is sampled at a rate $F_s > 2F_{max} = 2B$.
- $u_a(t)$ can be exactly recovered from its sample values:

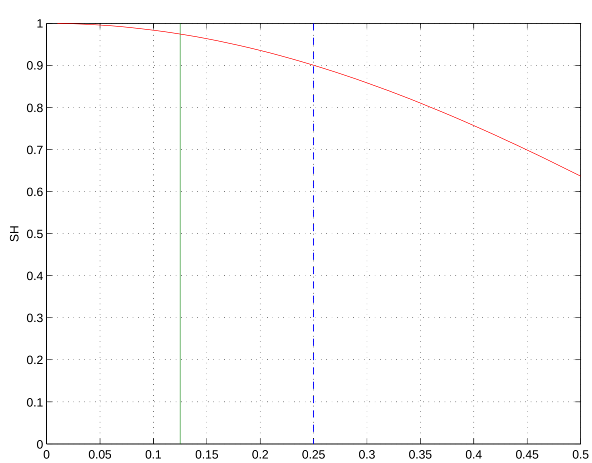
$$u_a(t) = \sum_{n=-\infty}^{\infty} u_a(nT_s) \frac{\sin\left\{\frac{\pi}{T_s}(t - nT_s)\right\}}{\frac{\pi}{T_s}(t - nT_s)}$$

- If $F_s = 2F_{\max}$, F_s is denoted by F_N, the Nyquist rate.
- Not causal: check $n > 0$
- To calculate analog signal at t, we need sampled values for all future times, corresponding to $n > 0$
- Hence for control purposes, this reconstruction is not useful.
- Provides absolute minimum limit of sampling rate.
- If sampling rate is lower than this minimum, no filter (whether causal or not) can achieve exact reproduction of the continuous function from the sampled signals.

10. Frequency Domain Interpretation of ZOH

Gain of Sample and Hold (see Text),

$$|SH| = \left|\frac{\sin \pi f}{\pi f}\right|$$



- Least distortion for $f \simeq 0$
- Max. distortion for $f \simeq 0.5$
- Recall normalized frequency:

$$f = F/F_s, \quad f_{\max} = B/F_s$$

- Minimum sampling rate

$$F_{s,1} = 2B \Rightarrow f_{\max,1} = 0.5$$

- Sample at twice the minimum

$$F_{s,2} = 4B \Rightarrow f_{\max,2} = 0.25$$

Maximum deviation = 10%

- Sample at 4 times minimum

$$F_{s,3} = 8B \Rightarrow f_{\max,3} = 0.125$$

Maximum deviation = 3%

- Fast sampling is better

11. Rules for Sampling Rate Selection

- Minimum sampling rate = twice band width
- Problems
  - Not really band limited
  - Systems are generally nonlinear
  - Shannon's reconstruction cannot be implemented, have to use ZOH
- Solution: sample faster
  - Number of samples in rise time = 4 to 10
  - Sample 10 to 30 times bandwidth
  - Use 10 times Shannon's sampling rate
  - $\omega_c T_s = 0.15$ to 0.5, where, ω_c = crossover frequency

12. Filtering - Motivation

- Measurements are often corrupted by high frequency noise, which have to be filtered before further processing.
- Systems that transmit the low frequency information while removing the effect of high frequency noise are known as *low pass filters* and this action is known as *low pass filtering*.
- Sometimes we are interested in monitoring of a transient response so as to take early corrective action. Often this requires a derivative action that works on the basis of the slope of the response curve. We will see later that this requires the usage of the high frequency content of the response.
- Indeed we may be interested in filtering of the frequency content in some arbitrary frequency range while passing the others.
- Will demonstrate that such things can be achieved by suitable choice of pole and zero locations.

13. Filter Design

- Apply input $u(k) = a^k 1(k)$ to an LTI system with with transfer function $G(z)$.
- Want to know what happens to frequency content of u by $G(z)$. Let a be of the form $e^{j\theta}$ and let $G(z)$ not have a pole at a.

$$Y(z) = G(z)\frac{z}{z-a}$$
$$Y(z) = e_0 + e_1\frac{z}{z-a} + \{\text{terms due to poles of } G(z)\}$$

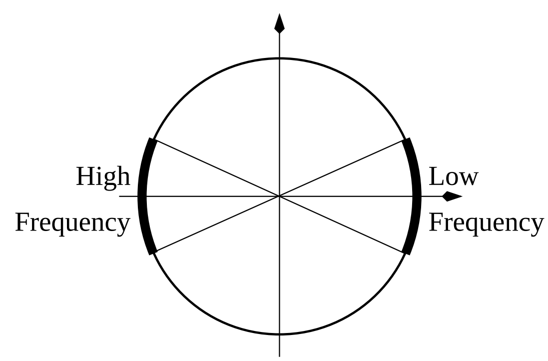
If e_1 is large, input is present in the output y. If e_1 is small, effect of u is removed.

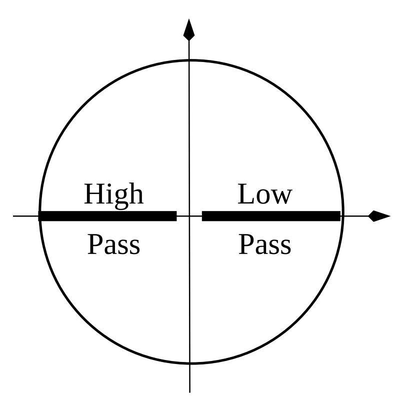
$$G(z)\frac{z}{z-a} = e_0 + e_1\frac{z}{z-a} + \{\text{ terms due to the poles of } G(z)\}$$
$$e_1 = \frac{z-a}{z}G(z)\frac{z}{z-a}\Big|_{z=a} = G(a)$$

- If we want to pass the input signal a^k in the output, choose $G(a)$ large
- A small $G(a)$ would result in the reduction of this input signal in the output
- Large $G(a)$ can be achieved if $G(z)$ has a pole close to a
- A zero of $G(z)$ near a will ensure the rejection of the effect of u on the output.

14. Approach to Filter Design

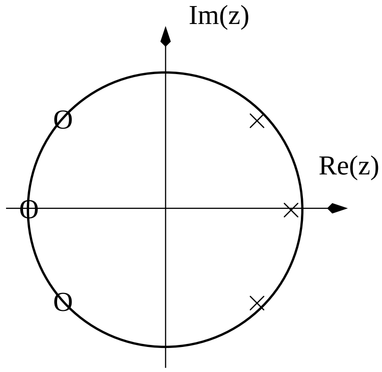
- If input frequency w_0 is to be passed through $G(z)$, place poles of $G(z)$ near w_0
- If it is to be filtered, place zeros of $G(z)$ near w_0
- Unique frequency values are in $(-\pi, \pi]$
- w close to 0 corresponds to low frequencies while w close to $\pm\pi$ corresponds to high frequencies
- Notice that e^{jw} with $w \in (-\pi, \pi]$ defines the unit circle. As a result, we can mark the low and high frequency regions as in the figure:



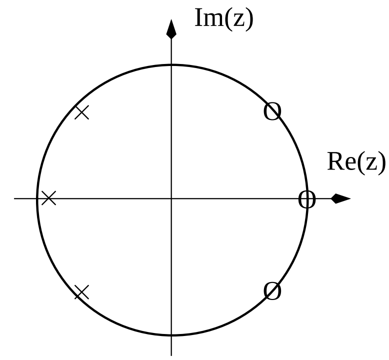


15. Low and High Pass Filters

- To pass signals of frequency w_0, we should place poles near w_0
- To reject w_0, we should place zeros near w_0



Low pass filter



High pass filter

- Place the poles inside unit circle for stability
- If complex, choose in conjugate pairs

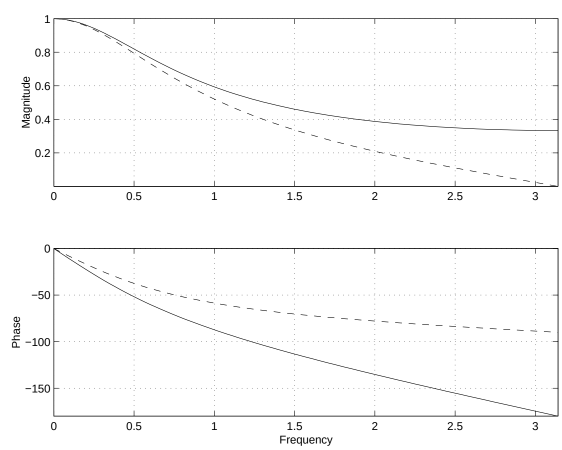
16. Low Pass Filter Example - 1

$$G_1(z) = \frac{0.5}{z - 0.5}$$

$G_1(z)|_{z=1} = 1$, so that its steady state gain is 1. Substituting $z = e^{jw}$, we get

$$\begin{aligned} G_1(e^{jw}) &= \frac{0.5}{e^{jw} - 0.5} \\ &= \frac{0.5}{(\cos w - 0.5) + j \sin w} \\ &= 0.5 \frac{(\cos w - 0.5) - j \sin w}{(\cos w - 0.5)^2 + \sin^2 w} \\ |G_1(e^{jw})| &= \frac{0.5}{\sqrt{1.25 - \cos w}} \\ \angle G_1(e^{jw}) &= -\tan^{-1}\left(\frac{\sin w}{\cos w - 0.5}\right) \end{aligned}$$

This filter magnifies the signal frequencies near $w = 0$ in relation to other frequencies



17. Low Pass Filter Example - 2

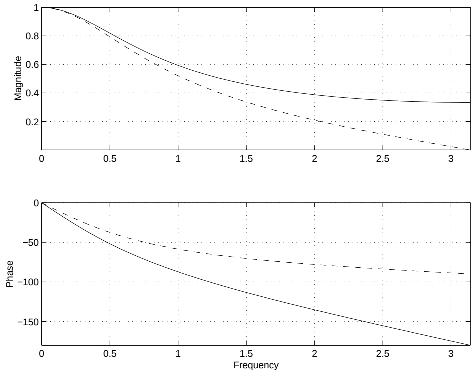
To G_1, add a zero at $z = -1$

$$G_2(z) = 0.25\frac{z+1}{z-0.5}$$

Notice that the factor 0.25 is included so as to make $|G_2(z)|_{z=1} = 1$.

$$\frac{G_2(e^{jw})}{K} = \frac{e^{jw}+1}{e^{jw}-0.5} = \frac{\cos w + j\sin w + 1}{\cos w + j\sin w - 0.5}$$
$$= \frac{[(\cos w + 1) + j\sin w][(\cos w - 0.5) - j\sin w]}{(\cos w - 0.5)^2 + \sin^2 w}$$
$$\frac{G_2(e^{jw})}{0.25} = \frac{(\cos^2 w + 0.5\cos w - 0.5) + \sin^2 w}{\sin^2 w + \cos^2 w + 0.25 - \cos w}$$
$$+ \frac{j\sin w(\cos w - 0.5 - \cos w - 1)}{\sin^2 w + \cos^2 w + 0.25 - \cos w}$$
$$G_2 = \frac{(0.5 + 0.5\cos w) - 1.5j\sin w}{1.25 - \cos w}0.25$$

- G_1 solid line, G_2 broken line
- $|G_2(e^{jw})| < |G_1(e^{jw})|\ \forall w > 0$. Thus, G_2 is a better low pass filter.



18. Low Pass Filter Example - 3

Calculate the response of $G_3(z) = \dfrac{z+1}{z-1}$ for input $u(n) = (-1)^n 1(n)$

$$G_3(z) = \frac{z}{z-1} + \frac{1}{z-1}$$
$$g_3(n) = 1(n) + 1(n-1)$$
$$y(n) = \sum_{i=-\infty}^{\infty} g(i)u(n-i)$$
$$= \sum_{i=-\infty}^{\infty} [1(i) + 1(i-1)]u(n-i)$$
$$= \sum_{i=0}^{\infty} u(n-i) + \sum_{i=1}^{\infty} u(n-i)$$
$$= 2\sum_{i=0}^{n} u(n-i) - u(n)$$
$$= \left[2\sum_{i=0}^{n}(-1)^{n-i}\right]1(n) - (-1)^n 1(n)$$

$$k = n - i \Rightarrow i = 0 \Rightarrow k = n,$$
$$i = n \Rightarrow k = 0$$

On simplifying,

$$y(n) = \left[2\sum_{k=n}^{0}(-1)^k\right]1(n) - (-1)^n 1(n)$$
$$= 2\frac{1-(-1)^{n+1}}{1-(-1)}1(n) - (-1)^n 1(n)$$
$$= 1(n)\left[1 - (-1)^{n+1} - (-1)^n\right]$$
$$= 1(n)$$

- This shows that $(-1)^n$ has been filtered.
- This is because the filter has a zero at $(-1, 0)$.

19. Fourier Transform of a Moving Average Filter - Example

$$y(n) = \frac{1}{3}[u(n+1) + u(n) + u(n-1)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} g(k)u(n-k)$$

$$= g(-1)u(n+1) + g(0)u(n) + g(1)u(n-1)$$

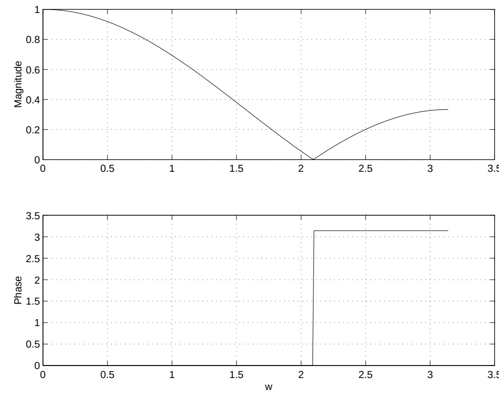
$$g(-1) = g(0) = g(1) = \frac{1}{3}$$

$$G\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} g(n)z^{-n}|_{z=e^{jw}}$$

$$= \frac{1}{3}\left(e^{jw} + 1 + e^{-jw}\right)$$

$$= \frac{1}{3}(1 + 2\cos w)$$

$$|G\left(e^{jw}\right)| = |\frac{1}{3}(1 + 2\cos w)|$$



$$Arg(G) = \begin{cases} 0 & 0 \le w < \frac{2\pi}{3} \\ \pi & \frac{2\pi}{3} \le w < \pi \end{cases}$$

```
w = 0:0.01:pi;
subplot(2,1,1)
plot(w,abs(1+2*cos(w))/3),
grid, ylabel('Magnitude')
subplot(2,1,2)
plot(w,angle(1+2*cos(w))),
grid, xlabel('w'), ylabel('Phase')
```


20. Fourier Transform of a Differencing Filter - Example

$$y(n) = u(n) - u(n-1)$$

$$g(0) = 1, \quad g(1) = -1$$

$$G\left(e^{jw}\right) = G(z)|_{z=e^{jw}}$$

$$= \sum_{n=-\infty}^{\infty} g(n)z^{-n}|_{z=e^{jw}}$$

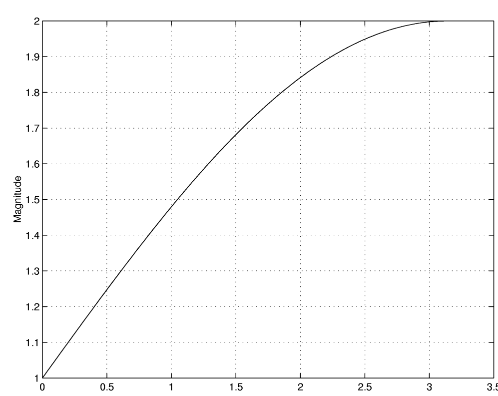
$$= 1 - e^{-jw}$$

$$= e^{-jw/2}\left(e^{jw/2} - e^{-jw/2}\right)$$

$$= 2je^{-jw/2}\sin\frac{w}{2}$$

$$|G| = 2|\sin\frac{w}{2}|$$

```
w = 0:0.01:pi;
plot(w,abs(1+sin(w/2))), grid, ...
xlabel('w'), ylabel('Magnitude')
```




```
sysd = tf([1 -1],1,-1);
w = logspace(-2,0.5);
bode(sysd,w)
```

- Fourier Transform pair for continuous signals:

$$x[t] = \int_{-\infty}^{\infty} X[F] e^{j2\pi Ft} dF \qquad X[F] = \int_{-\infty}^{\infty} x[t] e^{-j2\pi Ft} dt$$

- Fourier Transform pair for discrete time signals (DTFT):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad x(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega m} d\omega$$

- Discrete Fourier Transform (DFT):

$$G(k) = \sum_{n=0}^{N-1} g(n) e^{-j2\pi nk/N} \qquad g(n) = \frac{1}{N} \sum_{k=0}^{N-1} G(k) e^{j2\pi nk/N}$$

- Fast Fourier Transform (FFT): Fast method to implement DFT

