1. Noise and Prediction Models: ARMAX

ARIMAX model, with $\Delta = 1 - z^{-1}$:

$$\begin{aligned} Ay(n) &= Bu(n-k) + \frac{C}{\Delta}\xi(n) \\ A\Delta y(n) &= B\Delta u(n-k) + C\xi(n) \\ \hat{y}(n+j|n) &= \frac{E_j B\Delta}{C} u(n+j-k) + \frac{F_j}{C} y(n) \end{aligned}$$

ARIX model:

$$\begin{aligned} Ay(n) &= Bu(n-k) + \frac{1}{\Delta}\xi(n) \\ 1 &= E_j A\Delta + z^{-j} F_j \\ \hat{y}(n+j|t) &= E_j B\Delta u(n+j-k) + F_j y(n) \end{aligned}$$

2. Generalized Predictive Control: Index to Minimize

Generalized Predictive Control for ARIX Model:

$$A(z)y(n) = z^{-k}B(z)u(n) + \frac{1}{\Delta}\xi(n)$$

Optimization index to minimize:

$$\begin{aligned} J_{\text{GPC}} = {} & [\hat{y}(n+k) - r(n+k)]^2 + \cdots \\ & + [\hat{y}(n+k+N) - r(n+k+N)]^2 \\ & + \rho(\Delta u(n))^2 + \rho(\Delta u(n+1))^2 + \cdots + \rho(\Delta u^2(n+N))^2. \end{aligned}$$

- Use prediction model,
- determine $\hat{y}$
- substitute in the above and minimize
- to get an expression for u

3. Prediction Model for GPC

$$\underline{\hat{y}} = G\underline{u} + H_1\underline{u}_{\text{old}} + H_2\underline{y}_{\text{old}}$$

$$G\underline{u} = \begin{bmatrix} g_{k,0} & 0 & \cdots & 0 \\ g_{k+1,1} & g_{k+1,0} & \cdots & 0 \\ \vdots & & & \\ g_{k+N,N} & g_{k+N,N-1} & \cdots & g_{k+N,0} \end{bmatrix} \begin{bmatrix} \Delta u(n) \\ \Delta u(n+1) \\ \vdots \\ \Delta u(n+N) \end{bmatrix}$$

$$H_1\underline{u}_{\text{old}} = \begin{bmatrix} g_{k,1} & \cdots & g_{k,\mathrm{d}G_k} \\ g_{k+1,2} & \cdots & g_{k+1,\mathrm{d}G_{k+1}} \\ \vdots & & \\ g_{k+N,N+1} & \cdots & g_{k+N,\mathrm{d}G_{k+N}} \end{bmatrix} \begin{bmatrix} \Delta u(n-1) \\ \Delta u(n-2) \\ \vdots \\ \Delta u(n-k+1-\mathrm{d}B) \end{bmatrix}$$

$$H_2\underline{y}_{\text{old}} = \begin{bmatrix} f_{k,0} & \cdots & f_{k,\mathrm{d}A} \\ f_{k+1,0} & \cdots & f_{k+1,\mathrm{d}A} \\ \vdots & & \\ f_{k+N,0} & \cdots & f_{k+N,\mathrm{d}A} \end{bmatrix} \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-\mathrm{d}A) \end{bmatrix}$$

4. GPC Example

$$(1-0.8z^{-1})y(n) = z^{-1}(0.4+0.6z^{-1})u(n) + \frac{1}{\Delta}e(n)$$

$$A = 1-0.8z^{-1}, \quad B = 0.4+0.6z^{-1}, \quad k=1 \quad N=3$$

$$1 = E_j\Delta A + z^{-j}F_j, \quad G_j = E_jB$$

$$E_1 = 1$$

$$E_2 = 1+1.8z^{-1}$$

$$E_3 = 1+1.8z^{-1}+2.44z^{-2}$$

$$E_4 = 1+1.8z^{-1}+2.44z^{-2}+2.9520z^{-3}$$

$$F_1 = 1.8000-0.8000z^{-1}$$

$$F_2 = 2.4400-1.4400z^{-1}$$

$$F_3 = 2.9520-1.9520z^{-1}$$

$$F_4 = 3.3616-2.3616z^{-1}$$

5. GPC Example

$$G_1 = 0.4 + 0.60z^{-1}$$
$$G_2 = 0.4 + 1.32z^{-1} + 1.0800z^{-2}$$
$$G_3 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 1.4640z^{-3}$$
$$G_4 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 2.6448z^{-3} + 1.7712z^{-4}$$

$$G = \begin{bmatrix} g_{k,0} & 0 & \cdots & 0 \\ g_{k+1,1} & g_{k+1,0} & \cdots & 0 \\ \vdots & & & \\ g_{k+N,N} & g_{k+N,N-1} & \cdots & g_{k+N,0} \end{bmatrix}$$
$$= \begin{bmatrix} 0.4000 & 0 & 0 & 0 \\ 1.3200 & 0.4000 & 0 & 0 \\ 2.0560 & 1.3200 & 0.4000 & 0 \\ 2.6448 & 2.0560 & 1.3200 & 0.4000 \end{bmatrix}$$

6. GPC Example

$$G_1 = 0.4 + 0.60z^{-1}$$
$$G_2 = 0.4 + 1.32z^{-1} + 1.0800z^{-2}$$
$$G_3 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 1.4640z^{-3}$$
$$G_4 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 2.6448z^{-3} + 1.7712z^{-4}$$

$$H_1 = \begin{bmatrix} g_{k,1} & \cdots & g_{k,\mathrm{d}G_k} \\ g_{k+1,2} & \cdots & g_{k+1,\mathrm{d}G_{k+1}} \\ \vdots & & \\ g_{k+N,N+1} & \cdots & g_{k+N,\mathrm{d}G_{k+N}} \end{bmatrix} = \begin{bmatrix} 0.6000 \\ 1.0800 \\ 1.4640 \\ 1.7712 \end{bmatrix}$$

7. GPC Example

$$F_1 = 1.8000 - 0.8000z^{-1}$$
$$F_2 = 2.4400 - 1.4400z^{-1}$$
$$F_3 = 2.9520 - 1.9520z^{-1}$$
$$F_4 = 3.3616 - 2.3616z^{-1}$$
$$H_2 = \begin{bmatrix} f_{k,0} & \cdots & f_{k,\mathrm{d}A} \\ f_{k+1,0} & \cdots & f_{k+1,\mathrm{d}A} \\ \vdots & & \\ f_{k+N,0} & \cdots & f_{k+N,\mathrm{d}A} \end{bmatrix} = \begin{bmatrix} 1.8000 & -0.8000 \\ 2.4400 & -1.4400 \\ 2.9520 & -1.9520 \\ 3.3616 & -2.3616 \end{bmatrix}$$

8. GPC: Optimization Index

Contribution from optimization index is from two sources:

$$\begin{aligned} J_{\mathrm{GPC}} = {} & [\hat{y}(n+k) - r(n+k)]^2 + \cdots \\ & +[\hat{y}(n+k+N) - r(n+k+N)]^2 \\ & + \rho(\Delta u(n))^2 + \rho(\Delta u(n+1))^2 + \cdots + \rho(\Delta u^2(n+N))^2 \end{aligned}$$

- Contribution from error
- Control effort

Will begin with error terms

9. Setpoint Tracking

Recall prediction model and define setpoint:

$$\underline{\hat{y}} = G\underline{u} + H_1\underline{u}_{\text{old}} + H_2\underline{y}_{\text{old}}$$

$$\underline{r} = \begin{bmatrix} r(n+k) \\ \vdots \\ r(n+k+N) \end{bmatrix}$$

Subtract $\underline{r}$ from both sides of $\hat{y}$ equation, equate to zero

$$\underline{\hat{y}} - \underline{r} = G\underline{u} + H_1\underline{u}_{\text{old}} + H_2\underline{y}_{\text{old}} - \underline{r} - \underline{e} = 0$$

Reason for $\underline{e}$: it is a nonsquare system. Equivalent to

$$G\underline{u} = \underline{r} - H_1\underline{u}_{\text{old}} - H_2\underline{y}_{\text{old}} + \underline{e}$$

Can't make $\underline{e} \equiv 0$, Minimize its square, by suitably choosing $\underline{u}$:

$$\min_{u} J = \underline{e}^T\underline{e} = e^2(n+k) + \cdots + e^2(n+k+N) = \text{I Term}$$

10. Least Squares Solution: A Summary

Least squares problem. Solve the following with a tall G matrix:

$$G\underline{u} = \underline{b} + \underline{e}$$

- Cannot make $\underline{e} \equiv 0$, because, G is not square
- Can at most minimize the square of error terms

$$\min_{\underline{u}} \underline{e}^T\underline{e}$$

- Differentiate $\underline{e}^T\underline{e}$ with respect to $\underline{u}$ and equate it to zero
- Called least squares solution:

$$\underline{u} = (G^TG)^{-1}G^T\underline{b}$$

- Summary:
  To minimize $\underline{e}^T\underline{e}$, solve $G\underline{u} = \underline{b}$ in a least squares sense

11. Apply Least Squares Solution to GPC

Minimization of $\underline{e}^T\underline{e}$ is achieved in

$$G\underline{u} = \underline{b} + \underline{e}$$

by solving $G\underline{u} = \underline{b}$ in a least squares sense:

$$\underline{u} = (G^TG)^{-1}G^T\underline{b}$$

Recall GPC: minimize $\underline{e}^T\underline{e}$ in

$$G\underline{u} = \underline{r} - H_1\underline{u}_{\text{old}} - H_2\underline{y}_{\text{old}} + \underline{e}$$

Solve the following in least squares sense:

$$G\underline{u} = \underline{r} - H_1\underline{u}_{\text{old}} - H_2\underline{y}_{\text{old}}$$

Solution:

$$\begin{aligned}\underline{u} &= (G^TG)^{-1}G^T(\underline{r} - H_1\underline{u}_{\text{old}} - H_2\underline{y}_{\text{old}}) \\ &\triangleq K_1(\underline{r} - H_1\underline{u}_{\text{old}} - H_2\underline{y}_{\text{old}})\end{aligned}$$

12. GPC Control Law

To reduce control effort also,

$$\begin{aligned}\min_{e,u} J = {} & e^2(n+k) + \cdots + e^2(n+k+N) \\ & + \rho(\Delta u(n))^2 + \cdots + \rho(\Delta u(n+N_u))^2\end{aligned}$$

Differentiate w.r.t. Δu also, equate to 0

$$2\rho\Delta u(n+i) = 0, \;\forall i \in [0, N_u]$$

13. GPC Control Law - Continued

Minimizing with respect to both error and control effort,

$$\begin{bmatrix} G \\ \rho I \end{bmatrix} \underline{u} = \begin{bmatrix} \underline{r} - H_1 \underline{u}_{\text{old}} - H_2 \underline{y}_{\text{old}} \\ 0 \end{bmatrix}$$

The least squares solution to this problem is

$$\begin{aligned} \underline{u} &= \left(\begin{bmatrix} G^T & \rho I \end{bmatrix} \begin{bmatrix} G \\ \rho I \end{bmatrix} \right)^{-1} \begin{bmatrix} G^T & \rho I \end{bmatrix} \begin{bmatrix} \underline{r} - H_1 \underline{u}_{\text{old}} - H_2 \underline{y}_{\text{old}} \\ 0 \end{bmatrix} \\ &= (G^T G + \rho^2 I)^{-1} G^T (\underline{r} - H_1 \underline{u}_{\text{old}} - H_2 \underline{y}_{\text{old}}) \\ &= \begin{bmatrix} \Delta u(n) \\ \Delta u(n+1) \\ \vdots \\ \Delta u(n+N_u) \end{bmatrix} \end{aligned}$$

Implement only the first row, *i.e.*, $\Delta u(n)$, at every time instant

14. GPC Example - Continued

$$(1 - 0.8z^{-1}) y(n) = z^{-1}(0.4 + 0.6z^{-1}) u(n) + \frac{1}{\Delta} \xi(n)$$

$$G = \begin{bmatrix} 0.4000 & 0 & 0 & 0 \\ 1.3200 & 0.4000 & 0 & 0 \\ 2.0560 & 1.3200 & 0.4000 & 0 \\ 2.6448 & 2.0560 & 1.3200 & 0.4000 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0.6000 \\ 1.0800 \\ 1.4640 \\ 1.7712 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 1.8000 & -0.8000 \\ 2.4400 & -1.4400 \\ 2.9520 & -1.9520 \\ 3.3616 & -2.3616 \end{bmatrix}$$

$$\begin{aligned} \rho &= \sqrt{0.8} \\ \underline{u} &= K\underline{r} - KH_1 \underline{u}_{\text{old}} - KH_2 \underline{y}_{\text{old}} \\ K &= (G^T G + 0.8I)^{-1} G^T \end{aligned}$$

15. GPC Example - Control Law

$$\underline{u} = K\underline{r} - KH_1\underline{u}_{\text{old}} - KH_2\underline{y}_{\text{old}}$$

$$K = \begin{bmatrix} 0.1334 & 0.2864 & 0.1496 & -0.0022 \\ -0.1538 & -0.1968 & 0.1134 & 0.1496 \\ -0.0285 & -0.2155 & -0.1968 & 0.2864 \\ 0.0004 & -0.0285 & -0.1538 & 0.1334 \end{bmatrix}$$

$$KH_1 = \begin{bmatrix} 0.6045 \\ 0.1262 \\ -0.0307 \\ -0.0194 \end{bmatrix}, \quad KH_2 = \begin{bmatrix} 1.3732 & -0.8060 \\ 0.0807 & -0.1683 \\ -0.1953 & 0.0409 \\ -0.0744 & 0.0259 \end{bmatrix}$$

Control law is given by the first line:

$$\begin{aligned} \Delta u(n) = {} & 0.1334r(n+1) + 0.2864r(n+2) + 0.1496r(n+3) \\ & - 0.0022r(n+4) \\ & - 0.6045\Delta u(n-1) - 1.3732y(n) + 0.8060y(n-1) \end{aligned}$$

16. GPC Example - Code

```
% Camacho and Bordon's GPC example; Control law
%
A=[1 -0.8]; dA=1; B=[0.4 0.6]; dB=1; N=3; k=1;
[K,KH1,KH2] = gpc_bas(A,dA,B,dB,N,k,rho)

function [K,KH1,KH2] = gpc_bas(A,dA,B,dB,N,k,rho)
D=[1 -1]; dD=1; AD=conv(A,D); dAD=dA+1; Nu=N+k;
zj = 1; dzj = 0; G = zeros(Nu);
H1 = zeros(Nu,k-1+dB); H2 = zeros(Nu,dA+1);
for j = 1:Nu,
    zj = conv(zj,[0,1]); dzj = dzj + 1;
    [Fj,dFj,Ej,dEj] = xdync(zj,dzj,AD,dAD,1,0);
    [Gj,dGj] = polmul(B,dB,Ej,dEj);
    G(j,1:dGj) = flip(Gj(1:dGj));
    H1(j,1:k-1+dB) = Gj(dGj+1:dGj+k-1+dB);
    H2(j,1:dA+1) = Fj;
```

```
end
rho = 0.8;
K = inv(G'*G+rho*eye(Nu))*G';
KH1 = K * H1;
KH2 = K * H2;
```