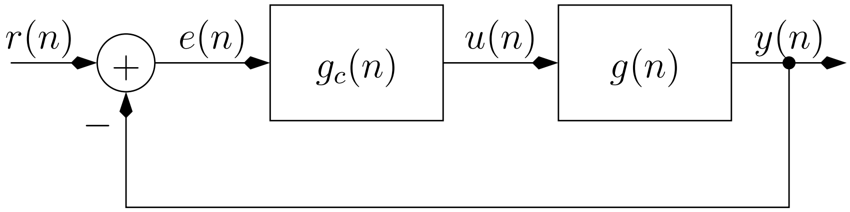
1. Internal Stability



- Notion in UG classes: output has to be stable
- Output being stable is not sufficient
- Every signal in the loop should be bounded
- If any signal is unbounded, will result in saturation / overflow / explosion
- When every signal is bounded, called internal stability
- If output is stable and if there is no unstable pole-zero cancellation, internal stability

2. Pole-Zero Cancellation: Init. Condition

Study the system

$$\begin{aligned} x(k+1) - ax(k) &= u(k+1) - au(k) \\ y(k) &= x(k) \end{aligned}$$

Can we take Z-transform

$$(z-a)X(z) = (z-a)U(z)$$

and cancel $z-a$ in

$$X(z) = \frac{z-a}{z-a}U(z) = U(z)$$

Because we have taken Z-transform, this is y_u

$$y_u(k) = u(k)$$

3. Pole-Zero Cancellation: Consider Initial Condition

$$x(k+1) - ax(k) = u(k+1) - au(k)$$
$$y(k) = x(k)$$

From the previous slide,

$$y_u = u(k)$$

$y_x \Rightarrow$ input = 0

$$x(k+1) = ax(k)$$
$$y_x(k) = x(k) = ax(k-1) = a^2x(k-2) = \cdots = a^kx(0)$$
$$y_x(k) = a^kx(0)$$

Total solution:

$$y(k) = y_x(k) + y_u(k) = a^kx(0) + u(k)$$

4. Pole-Zero Cancellation: Can we take $x(0) = 0$?

$$y(k) = a^kx(0) + u(k)$$

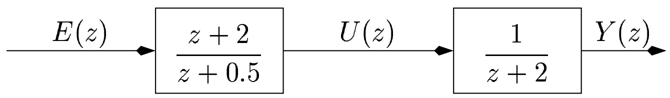
If $x(0) = 0$,

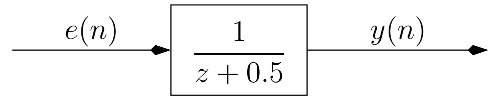
$$y(k) = \cancel{a^k}\cancel{x}(0) + u(k)$$

obtained by cancellation: If $x(0) = 0$, can cancel. Is it ok?

- Computers: zero = $\simeq 10^{-17}$
- Hardware: zero = μ voltages $\neq 0$
- Actual: $x(0) = \varepsilon$, a small no. $\neq 0$
- When $|a| > 1$, cancellation results in $\infty \times 0 = 0$, A blunder!
- Unstable pole-zero cancellation forbidden
- When $|a| < 1$, asymptotically correct, so acceptable

5. Unstable Pole-Zero Cancelled System





Solution of cancelled system:

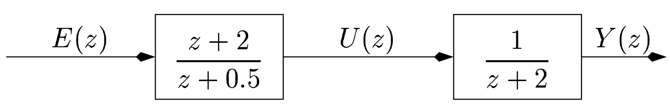
$$
\begin{aligned}
x_s(k+1) &= -0.5x_s(k) + e(k) \\
y(k)_s &= x_s(k)
\end{aligned}
$$

Iteratively solving,

$$y_s(k) = (-0.5)^m x_s(0) + \sum_{m=0}^{k-1} (-0.5)^m e(k-m-1)$$

Stable.

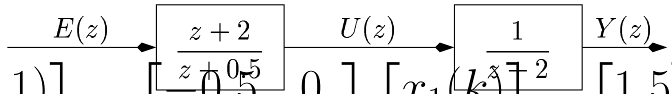
6. Complete System without Pole-Zero Cancellation



Can verify for first block:

$$
\begin{aligned}
\text{First block:} \quad x_1(k+1) &= -0.5x_1(k) + 1.5e(k) \\
u(k) &= x_1(k) + e(k) \\
\text{Second block:} \quad x_2(k+1) &= -2x_2(k) + u(k) \\
y(k) &= x_2(k) \\
\text{Substituting,} \quad x_2(k+1) &= -2x_2(k) + x_1(k) + e(k) \\
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} -0.5 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} e(k) \\
y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}
\end{aligned}
$$

7. Complete System without Pole-Zero Cancellation



$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} e(k)$$

$$\begin{aligned} y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &= CA^k x(0) + \sum_{m=0}^{k-1} CA^m b e(k-m-1) \end{aligned}$$

Can show that this is equal to

$$\frac{3}{2} \left[(-0.5)^k \ \ (-2)^k \right] \begin{bmatrix} x_1(0) \\ -x_1(0) + 1.5 x_2(0) \end{bmatrix} + \sum_{m=0}^{k-1} (-0.5)^m e(k-m-1)$$

$x(0) \neq 0$, results in $y(k)$ being unbounded! Whence -2?

8. Diagonalization

A: square matrix λ_i: ith eigenvalue x_i: ith eigenvector

$$\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ &\vdots \\ Ax_n &= \lambda_n x_n \end{aligned}$$

Stacking these side by side,

$$A \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$AS = S\Lambda$$

Assume the eigenvectors to be independent $\Rightarrow S^{-1}$ exists

$$A = S \Lambda S^{-1}$$

9. Diagonalization

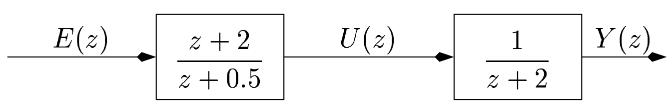
$$
\begin{aligned}
A &= S\Lambda S^{-1} \\
A^2 &= S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1} \\
A^3 &= S\Lambda^2 S^{-1} S\Lambda S^{-1} = S\Lambda^3 S^{-1} \\
&\vdots \\
A^k &= S\Lambda^k S^{-1}
\end{aligned}
$$

Easy to evaluate RHS:

$$
\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}
$$

This approach is used to arrive at the solution.

10. Condition for Cancellation of Poles and Zeros



Solution of system after cancellation:

$$
y_s(k) = (-0.5)^m x_s(0) + \sum_{m=0}^{k-1} (-0.5)^m e(k-m-1)
$$

Solution of system if there is no cancellation:

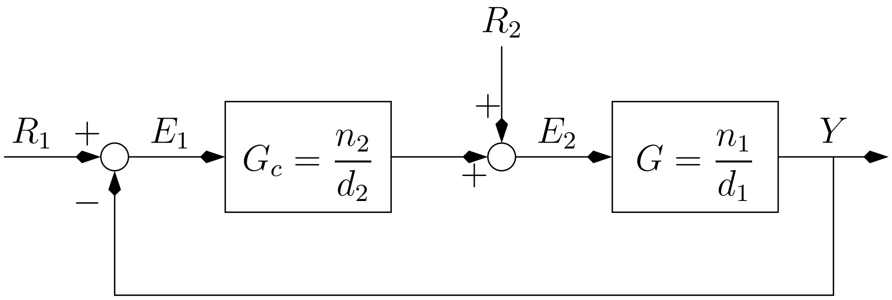
$$
\frac{3}{2}\left[(-0.5)^k \;\; (-2)^k\right] \begin{bmatrix} x_1(0) \\ -x_1(0) + 1.5x_2(0) \end{bmatrix} + \sum_{m=0}^{k-1} (-0.5)^m e(k-m-1
$$

Two solutions are identical only if

- $x_1(0) = 1.5x_2(0)$ or
- $x_1(0) = x_2(0) = 0$.

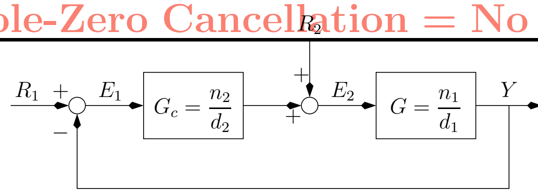
11. Forbid Unstable Pole-Zero Cancellation: Loop Variable

- Internal stability = all variables in loop are bounded for bounded external inputs at all locations
- Can be checked by the following closed loop diagram



- Can show that Output is stable + no pole-zero cancellation = internal stability

12. Unstable Pole-Zero Cancellation = No Internal Stability



$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+GG_c} & -\dfrac{G}{1+GG_c} \\ \dfrac{G_c}{1+G_cG} & \dfrac{1}{1+G_cG} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{d_1d_2}{n_1n_2+d_1d_2} & -\dfrac{n_1d_2}{n_1n_2+d_1d_2} \\ \dfrac{n_2d_1}{n_1n_2+d_1d_2} & \dfrac{d_1d_2}{n_1n_2+d_1d_2} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

Suppose d_1, n_2 have a common factor:

$$d_1 = (z+a)d_1'$$
$$n_2 = (z+a)n_1'$$

13. Unstable Pole-Zero Cancellation = No Internal Stability

Assume the cancellation of $z+a$,

$$G(z) = \frac{n_1(z)}{(z+a)d_1'(z)}, \qquad G_c(z) = \frac{(z+a)n_2'(z)}{d_2(z)}$$

with $|a| > 1$. Assume stability of

$$T_E = \frac{1}{1+GG_c} = \frac{d_1'd_2}{d_1'd_2 + n_1'n_2}$$

T.F. between R_2 and Y can be shown to be unstable. Let $R_1 = 0$.

$$\frac{Y}{R_2} = \frac{G}{1+GG_c} = \frac{n_1 d_2}{(d_1'd_2 + n_1 n_2')(z+a)}$$

It is unstable and a bounded signal injected at R_2 will produce an unbounded signal at Y.

14. Forbid Unstable Pole-Zero Cancellation, Get Causality

Not possible to realize this controller:

$$G_c = \frac{1+z^{-1}}{z^{-1}}$$

All sampled systems have at least one delay:

$$G(z^{-1}) = z^{-k}\frac{B(z^{-1})}{A(z^{-1})} = z^{-k}\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots}$$

- Controller not realizable $\Rightarrow$ there is a common factor z^{-1} between plant and controller
- $z^{-1} = 0 \Rightarrow z = \infty$, an unstable pole
- If unstable pole-zero cancellation is forbidden while designing controllers, z^{-1} cannot appear in the denominator of the controller - *i.e.*, controller is realizable

15. Delay Specification for Realizability

Closed loop delay has to be $\geq$ open loop delay:

$$G(z) = z^{-k}\frac{B(z)}{A(z)} = z^{-k}\frac{b_0 + b_1 z^{-1} + \cdots}{1 + a_1 z^{-1} + \cdots}$$

$b_0 \neq 0$. Suppose that we use a feedback controller of the form

$$G_c(z) = z^{-d}\frac{S(z)}{R(z)} = z^{-d}\frac{s_0 + s_1 z^{-1} + \cdots}{1 + r_1 z^{-1} + \cdots}$$

with $s_0 \neq 0$ and $d \geq 0$. Closed loop transfer function:

$$T = \frac{GG_c}{1+GG_c} = z^{-k-d}\frac{b_0 s_0 + (b_0 s_1 + b_1 s_0)z^{-1} + \cdots}{1 + (s_1 + r_1)z^{-1} + \cdots + z^{-k-d}(b_0 s_0 + \cdots)}$$

- Closed loop delay $= k + d \geq k =$ open loop delay.
- Can make it less only by $d < 0$, but controller is unrealizable.

16. Example: Closed Loop Delay Specification

Design a controller for the plant

$$G = z^{-2}\frac{1}{1 - 0.5z^{-1}}$$

so that the overall system has smaller delay:

$$T = z^{-1}\frac{1}{1 - az^{-1}}$$

Recall the standard closed loop transfer function:

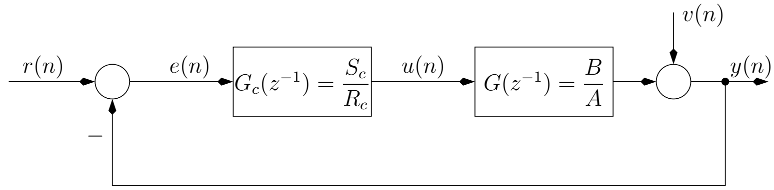
$$T = \frac{GG_c}{1+GG_c} \Rightarrow$$

Solving for G_c, and substituting for T, G

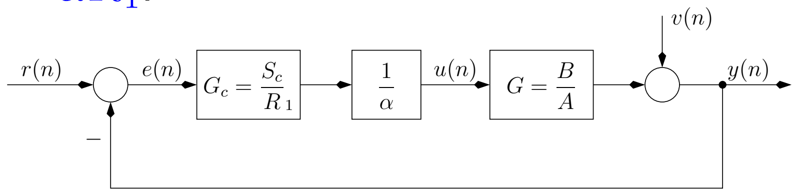
$$G_c = \frac{1}{G}\frac{T}{1-T} = \frac{1}{z^{-1}}\frac{1 - 0.5z^{-1}}{1 - (a+1)z^{-1}}$$

This controller is unrealizable, no matter what a is.

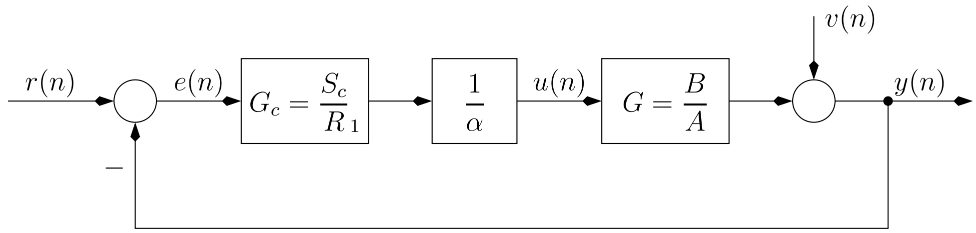
17. Internal Model Principle



- $\alpha(z^{-1})$ = least common denominator of the unstable poles of $R(z^{-1})$ and of $V(z^{-1})$
- Let there be no common factors between $\alpha(z^{-1})$ and $B(z^{-1})$
- Can find a controller $G_c(z)$ for servo/tracking (following R) and regulation (rejection of disturbance V) if R_c contains α, say, $R_c = \alpha R_1$:



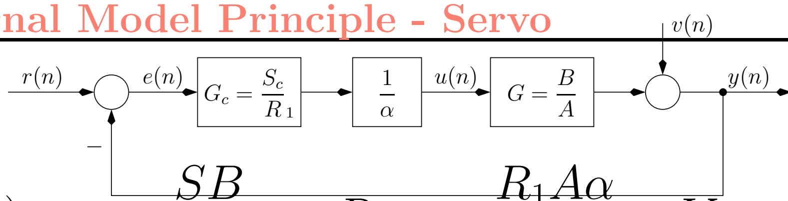
18. Internal Model Principle - Regulation



$$
\begin{aligned}
Y(z^{-1}) &= \frac{\dfrac{S}{R_1}\dfrac{1}{\alpha}\dfrac{B}{A}}{1 + \dfrac{S}{R_1}\dfrac{1}{\alpha}\dfrac{B}{A}} R + \frac{1}{1 + \dfrac{S}{R_1}\dfrac{1}{\alpha}\dfrac{B}{A}} V \\
&= \frac{SB}{R_1 A\alpha + SB} R + \frac{R_1 A\alpha}{R_1 A\alpha + SB} \frac{b_V}{\alpha a_V}
\end{aligned}
$$

- Unstable pole present in α gets cancelled
- Regulation problem verified

19. Internal Model Principle - Servo



$$Y(z^{-1}) = \frac{SB}{R_1 A\alpha + SB} R + \frac{R_1 A\alpha}{R_1 A\alpha + SB} V$$

Servo problem: assume $V = 0$:

$$\begin{aligned} E(z^{-1}) &= R(z^{-1}) - Y(z^{-1}) \\ &= \left(1 - \frac{SB}{R_1 A\alpha + SB}\right) R = \frac{R_1 A\alpha}{R_1 A\alpha + SB} \frac{b_V}{\alpha a_V} \end{aligned}$$

- Unstable poles of R are cancelled by zeros of α.
- Can choose R and S such that $R_1 A\alpha + SB$ has roots within the unit circle (pole placement)
- IM Principle: unstable poles of V, R appear in loop thro' α