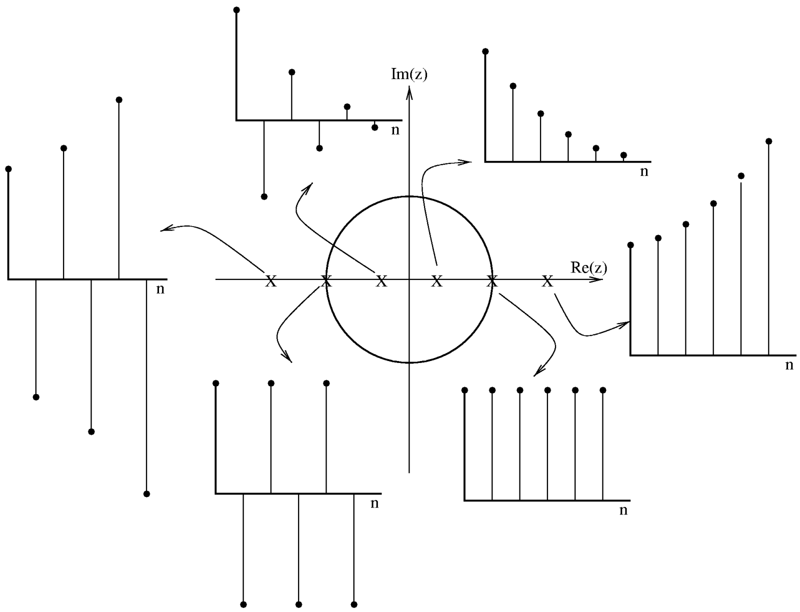
1. Nature of Impulse Response - Pole on Real Axis

Causal system transfer function:

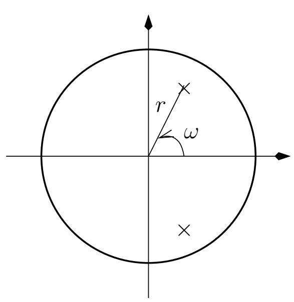
$$H(z) = \frac{z}{z-r} \quad \Rightarrow y(z) = \frac{z}{z-r} \quad \Rightarrow y(n) = r^n$$



- $r > 1$: the response grows monotonically
- $1 > r > 0$: y decays to zero monotonically
- $0 \geq r > -1$: oscillatory, decaying exponential
- $r < -1$: the output grows with oscillations

2. Nature of Impulse Response - Complex Conjugate Poles

$$H(z) = \frac{z^2}{(z - re^{j\omega})(z - re^{-j\omega})}$$



Calculate impulse response $Y(z)$:

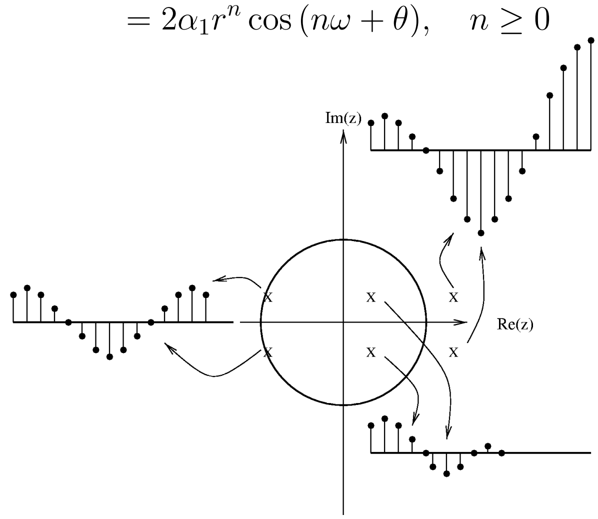
$$\frac{Y(z)}{z} = \frac{z}{(z - re^{j\omega})(z - re^{-j\omega})} = \frac{A}{z - re^{j\omega}} + \frac{A^*}{z - re^{-j\omega}}$$

A is complex, A^* is conjugate

$$y(n) = Ar^n e^{jn\omega} + A^* r^n e^{-jn\omega}$$

$$A = \alpha_1 e^{j\theta}$$

$$y(n) = \alpha_1 r^n \left[e^{j(n\omega+\theta)} + e^{-j(n\omega+\theta)} \right] = 2\alpha_1 r^n \cos(n\omega + \theta), \quad n \geq 0$$



$$\{u(n)\} = \sum_{k=-\infty}^{\infty} \{\delta(n-k)\} u(k) = \delta(n) * u(n)$$

Arbitrary input = sum of impulse functions

Corresponding response = sum of above sinusoids

3. Properties of Continuous Sinusoidal Signals

$$u_a(t) = A\cos(\Omega t + \theta), \quad -\infty < t < \infty$$

A: amplitude
Ω: frequency in rad/s
θ: phase in rad
F: frequency in $cycles/s$ or $Hertz$

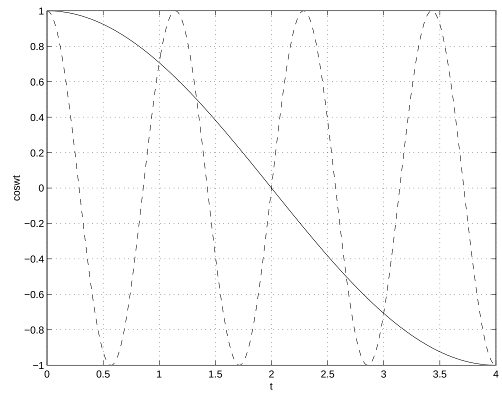
$$\Omega = 2\pi F$$
$$u_a(t) = A\cos(2\pi F t + \theta)$$
$$T_p = \frac{1}{F}$$

1. F fixed: periodic with period T_p

$$\begin{aligned} u_a[t + T_p] &= A\cos(2\pi F(t + \frac{1}{F}) + \theta) \\ &= A\cos(2\pi + 2\pi F t + \theta) \\ &= A\cos(2\pi F t + \theta) = u_a[t] \end{aligned}$$

2. Continuous signals with different frequencies are different

$$u_1 = \cos\left(2\pi\frac{t}{8}\right), \; u_2 = \cos\left(2\pi\frac{7t}{8}\right)$$



3. Frequency of $u \downarrow \Rightarrow$ rate of oscillation of signal $\downarrow$
   - t is a continuous variable

4. Complex Sinusoids

$$u_a[t] = Ae^{j(\Omega t + \theta)} = A[\cos(\Omega t + \theta) + j\sin(\Omega t + \theta)]$$

- For convenience, negative frequency
- Positive frequency = counter clockwise rotation
- Negative frequency = clockwise rotation

$$u_{a_1}[t] = A\cos(\Omega t + \theta) = \frac{A}{2}\left[e^{j(\Omega t + \theta)} + e^{-j(\Omega t + \theta)}\right] = Re\left[Ae^{j(\Omega t + \theta)}\right]$$

Re - real part. Im - imaginary

$$u_{a_2}[t] = A\sin(\Omega t + \theta) = Im\left[Ae^{j(\Omega t + \theta)}\right]$$

5. Properties of Discrete Sinusoidal Signals - Periodicity

$$u(n) = A\cos(wn + \theta), \; -\infty < n < \infty$$
$$w = 2\pi f$$

n integer variable, sample number
A amplitude of the sinusoid
w frequency in radians per sample
θ phase in radians.
f normalized frequency, cycles/sample

$u(n)$ is periodic with period N, $N > 0$, if and only if (iff)

$$u(n+N) = u(n) \; \forall n$$

The smallest nonzero N = fundamental period

$$u(n) = \cos(2\pi f_0 n + \theta)$$
$$u(n+N) = \cos(2\pi f_0 (n+N) + \theta)$$

Equal iff there exists an integer k:

$$2\pi f_0 N = 2k\pi \quad \Rightarrow f_0 = \frac{k}{N}$$

- f_0 is rational
- N obtained after cancelling the common factors in $\dfrac{k}{f_0}$ is known as fundamental period

A discrete time sinusoid is periodic only if its frequency f is a rational number

6. Properties of Discrete Sinusoids - Identical Signals

$$u(n) = A\cos(wn + \theta), \; -\infty < n < \infty$$
$$w = 2\pi f$$

n integer variable, sample number
A amplitude of the sinusoid
w frequency in radians per sample
θ phase in radians.
f normalized frequency, cycles/sample

Discrete time sinusoids whose frequencies are separated by integer multiple of 2π are identical, *i.e.*,

$$\cos((w_0 + 2\pi)n + \theta) = \cos(w_0 n + \theta), \; \forall n$$

All sinusoidal sequences $u_k(n)$

$$u_k(n) = A\cos(w_k n + \theta),$$
$$w_k = w_0 + 2k\pi, \; -\pi < w_0 < \pi$$

- are indistinguishable or identical
- Only the sinusoids in range $-\pi < w_0 < \pi$ are different

$$-\pi < w_0 < \pi \text{ or } -\frac{1}{2} < f_0 < \frac{1}{2}$$

This property is different from the previous one

- Now fixed n & varying f
- Previously fixed f & varying n

7. Properties of Discrete Sinusoids - Highest Frequency

$$u(n) = A\cos(wn+\theta), \; -\infty < n < \infty$$
$$w = 2\pi f$$

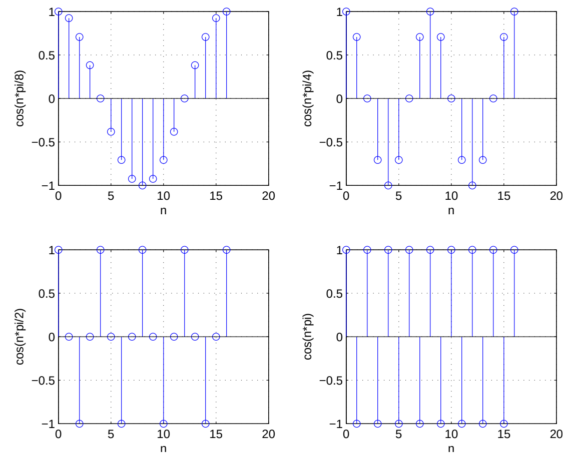
n	integer variable, sample number
A	amplitude of the sinusoid
w	frequency in radians per sample
θ	phase in radians.
f	normalized frequency (cycles/sample)

Highest oscillation

- $w = \pi$ or $w = -\pi$
- $f = \dfrac{1}{2}$ or $f = -\dfrac{1}{2}$

$\cos w_0 n$ has been plotted for

- w_0 values of $\dfrac{\pi}{8}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \pi$
- integer n



8. Sampling a Continuous Signal - Preliminaries

Let analog signal $u_a[t]$ have a frequency of F Hz

$$u_a[t] = A\cos(2\pi F t + \theta)$$

Uniform sampling rate (T_s s) or frequency (F_s Hz)

$$t = nT_s = \frac{n}{F_s}$$

$$\begin{aligned} u(n) &= u_a[nT_s] \quad -\infty < n < \infty \\ &= A\cos(2\pi F T_s n + \theta) \\ &= A\cos\left(2\pi \frac{F}{F_s} n + \theta\right) \end{aligned}$$

In our standard notation,

$$u(n) = A\cos(2\pi f n + \theta)$$

It follows that

$$f = \frac{F}{F_s} \qquad w = \frac{\Omega}{F_s} = \Omega T_s$$

Apply the uniqueness condition for sampled signals

$$-\frac{1}{2} < f < \frac{1}{2} \qquad -\pi < w < \pi$$

$$F_{\max} = \frac{F_s}{2} \qquad \Omega_{\max} = 2\pi F_{\max} = \pi F_s = \frac{\pi}{T_s}$$

9. Properties of Discrete Sinusoids - Alias

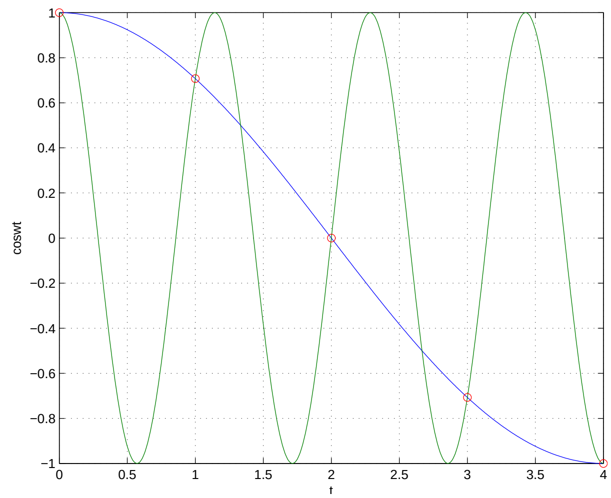
- As $w_0 \uparrow$, freq. of oscillation $\uparrow$, reaches maximum at $w_0 = \pi$
- What if $w_0 > \pi$?

$$
\begin{aligned}
w_1 &= w_0 \\
w_2 &= 2\pi - w_0 \\
u_1(n) &= A\cos w_1 n \\
&= A\cos w_0 n \\
u_2(n) &= A\cos w_2 n \\
&= A\cos(2\pi - w_0)n \\
&= A\cos w_0 n \\
&= u_1(n)
\end{aligned}
$$

w_2 is an alias of w_1

$$u_1[t] = \cos\left(2\pi\frac{t}{8}\right),\; u_2[t] = \cos\left(2\pi\frac{7t}{8}\right),\; T_s = 1$$

$$
\begin{aligned}
u_2(n) &= \cos\left(2\pi\frac{7n}{8}\right) = \cos 2\pi\left(1 - \frac{1}{8}\right)n \\
&= \cos\left(2\pi - \frac{2\pi}{8}\right)n = \cos\left(\frac{2\pi n}{8}\right) = u_1(n)
\end{aligned}
$$



10. Fourier Series of Continuous Periodic Signals

- $x(t)$ is periodic with a fundamental period $T_p = \frac{1}{F_0}$
- It has a Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_0 t}$$

- Want to calculate C_k
- Multiply both sides by $e^{-j2\pi l F_0 t}$
- Integrate from t_0 to $t_0 + T_p$, $T_p = \frac{1}{F_0}$

$$
\begin{aligned}
\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi l F_0 t}dt &= \int_{t_0}^{t_0+T_p} e^{-j2\pi l F_0 t}\left(\sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_0 t}\right)dt \\
&= \sum_{k=-\infty}^{\infty} C_k \int_{t_0}^{t_0+T_p} e^{j2\pi(k-l)F_0 t}dt
\end{aligned}
$$

11. Fourier Series of Continuous Periodic Signals

$$\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0t}dt = \sum_{k=-\infty}^{\infty} C_k \int_{t_0}^{t_0+T_p} e^{j2\pi(k-l)F_0t}dt$$

$$= C_lT_p + \sum_{k=-\infty,k\neq l}^{\infty} C_k \left.\frac{e^{j2\pi(k-l)F_0t}}{j2\pi(k-l)F_0}\right|_{t_0}^{t_0+T_p}$$

If $k-l\neq 0$, let $n=k-l$:

$$\left.e^{j2\pi nF_0t}\right|_{t_0}^{t_0+T_p} = e^{j2\pi nF_0(t_0+T_p)} - e^{j2\pi nF_0t_0} = e^{j2\pi nF_0t_0}(e^{j2\pi n}-1) = 0$$

Hence

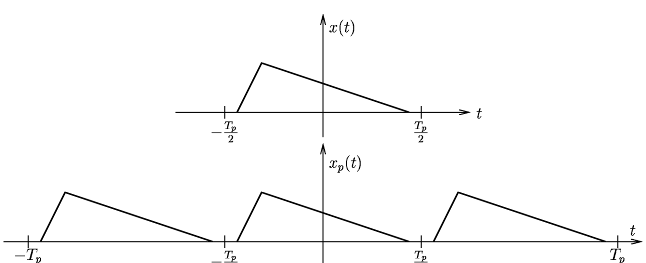
$$\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0t}dt = C_lT_p \Rightarrow$$

$$C_l = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0t}dt = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi lF_0t}dt$$

Since, periodic. $x(t)$, C_l: Fourier Series Pair

12. Fourier Transform of Continuous Aperiodic Signals

- Aperiodic $\Rightarrow$ no Fourier series
- If $x(t)=0$ outside $(-\frac{T_p}{2},\frac{T_p}{2})$, Construct a periodic signal $x_p[t]$ with a period T_p. $\lim_{T_p\to\infty} x_p[t] = x(t)$



$$x_p[t] = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kF_0t}, \quad F_0 = \frac{1}{T_p}$$

$$C_k = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x_p[t]e^{-j2\pi kF_0t}dt$$

As $x_p = x$ over one period,

$$C_k = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t)e^{-j2\pi kF_0t}dt$$

As x vanishes outside one period,

$$C_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)e^{-j2\pi kF_0t}dt$$

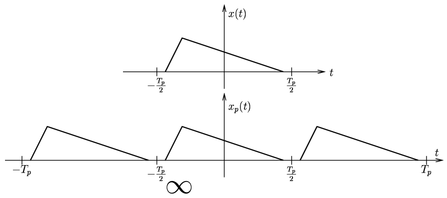
Define Fourier Transform of $x(t)$ as

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

$X(F)$ is a function of the continuous variable F. C_k are samples of $X(F)$.

$$C_k = \frac{1}{T_p}X[kF_0] = F_0X[kF_0]$$

13. Fourier Transform of Aperiodic Signals - Continued



$$x_p[t] = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_0 t}$$

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt$$

$$x_p[t] = F_0 \sum_{k=-\infty}^{\infty} X[kF_0] e^{j2\pi k F_0 t}$$

$$\Delta F \triangleq \frac{1}{T_p} = F_0$$

$$x_p[t] \triangleq \sum_{k=-\infty}^{\infty} X[k\Delta F] e^{j2\pi k \Delta F t} \Delta F$$

$$\begin{aligned} x(t) &= \lim_{T_p \to \infty} x_p[t] \\ &= \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X[k\Delta F] e^{j2\pi k \Delta F t} \Delta F \\ &= \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF \end{aligned}$$

If we let radian frequency $\Omega = 2\pi F$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X[\Omega] e^{j\Omega t} d\Omega,$$

$$X[\Omega] = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$x(t)$, $X[\Omega]$ are Fourier Transform Pair

14. Frequency Response ⇒ Discrete Time Fourier Transform

Apply $u(n) = e^{jwn}$ to $g(n)$ and obtain output y:

$$\begin{aligned} y(n) = g(n) * u(n) &= \sum_{k=-\infty}^{\infty} g(k) u(n-k) = \sum_{k=-\infty}^{\infty} g(k) e^{jw(n-k)} \\ &= e^{jwn} \sum_{k=-\infty}^{\infty} g(k) e^{-jwk} \end{aligned}$$

Define Discrete Time Fourier Transform

$$G(e^{jw}) \triangleq \sum_{k=-\infty}^{\infty} g(k) e^{-jwk} = \left. \sum_{k=-\infty}^{\infty} g(k) z^{-k} \right|_{z=e^{jw}} = G(z)|_{z=e^{jw}}$$

Provided, absolute convergence:

$$\sum_{k=-\infty}^{\infty} |g(k) e^{-jwk}| < \infty \Rightarrow \sum_{k=-\infty}^{\infty} |g(k)| < \infty$$

For causal systems, BIBO stability

15. Frequency Response - Continued

$$y(n) = e^{jwn} \sum_{k=-\infty}^{\infty} g(k) e^{-jwk} = e^{jwn} G(e^{jw})$$

Write in polar coordinates: $G(e^{jw}) = |G(e^{jw})| e^{j\varphi}$ - φ is phase angle:

$$y(n) = e^{jwn} |G(e^{jw})| e^{j\varphi} = |G(e^{jwn})| e^{j(wn+\varphi)}$$

1. Input is sinusoid $\Rightarrow$ output also is a sinusoid with following properties:
   - Output amplitude gets multiplied by the magnitude of $G(e^{jw})$
   - Output sinusoid shifts by φ with respect to input
2. At ω where $|G(e^{jw})|$ is large, the sinusoid gets amplified and at ω where it is small, the sinusoid gets attenuated.
   - The system with large gains at low frequencies and small gains at high frequencies are called low pass filters
   - Similarly high pass filters