1. Noise and Prediction Models: ARMAX

ARMAX Model:

$$Ay(n) = Bu(n-k) + C\xi(n)$$
$$C = E_j A + z^{-j} F_j$$
$$\hat{y}(n+j|t) = \frac{E_j B}{C} u(n+j-k) + \frac{F_j}{C} y(n)$$

ARIMAX model, with $\Delta = 1 - z^{-1}$:

$$Ay(n) = Bu(n-k) + \frac{C}{\Delta}\xi(n)$$
$$A\Delta y(n) = B\Delta u(n-k) + C\xi(n)$$
$$\hat{y}(n+j|n) = \frac{E_j B \Delta}{C} u(n+j-k) + \frac{F_j}{C} y(n)$$

2. Different Noise and Prediction Models: ARIX

ARIX model:

$$Ay(n) = Bu(n-k) + \frac{1}{\Delta}\xi(n)$$
$$1 = E_j A\Delta + z^{-j} F_j$$
$$\hat{y}(n+j|t) = E_j B\Delta u(n+j-k) + F_j y(n)$$

3. What are degrees of E_j, F_j?

$$\frac{C}{A} = E_j + z^{-j}\frac{F_j}{A}$$

$$
\begin{array}{r|llll}
 & 1 + 1.1z^{-1} & & & \\
\hline
1 - 0.6z^{-1} - 0.16z^{-2} & 1 & +0.5z^{-1} & & \\
 & 1 & -0.6z^{-1} & -0.16z^{-2} & \\
\hline
 & & +1.1z^{-1} & +0.16z^{-2} & \\
 & & +1.1z^{-1} & -0.66z^{-2} & -0.176z^{-3} \\
\hline
 & & & +0.82z^{-2} & +0.176z^{-3}
\end{array}
$$

$$\frac{1 + 0.5z^{-1}}{1 - 0.6z^{-1} - 0.16z^{-2}} = (1 + 1.1z^{-1}) + z^{-2}\frac{0.82 + 0.176z^{-1}}{1 - 0.6z^{-1} - 0.16z^{-2}}$$

4. Minimum Variance Control - A Summary

- Obtain a prediction model
- Try to minimize the effect of noise at k itself
- $y(n+k)$ = Past terms + Future noise terms
- By letting future noise terms to zero, obtain estimate
- $\hat{y}(n+k|n)$ = Past terms
- Equate this to zero and obtain controller
- ARMAX Model: $u(n) = -\frac{F_k}{E_k B} y(n)$
- ARIX Model: $\Delta u(n) = -\frac{F_k}{E_k B} y(n)$
- B is plant numerator. Recall $G = z^{-k}B/A$
- If G is of nonminimum phase, u will have unbounded signals

5. Generalized Minimum Variance Control

$$y(n+k) = \frac{E_k B}{C} u(n) + \frac{F_k}{C} y(n) + E_k \xi(n+k)$$

GMV - Minimize the variations in y at k itself, constraining u:

$$J = \mathscr{E}[(y(n+k) - \gamma r(n+k))^2 + \rho u^2(n)]$$
$$= \mathscr{E}[(\frac{E_k B u(n) + F_k y(n)}{C} + E_k \xi(n+k) - \gamma r(n+k))^2 + \rho u^2(n)]$$

As $\xi(n+k)$ is not correlated with $u(n)$ and $y(n)$, we get

$$= \mathscr{E}\left[(\frac{E_k B u + F_k y}{C} - \gamma r(n+k))^2 + \rho u^2(n)\right] + \mathscr{E}\left(E_k \xi(n+k)\right)^2$$

Differentiate with respect to u and equate to zero. II term $= 0$

6. Generalized Minimum Variance Control

Recall objective function to minimize:

$$J = \mathscr{E}\left[(\frac{E_k B u(n) + F_k y(n)}{C} - \gamma r(n+k))^2 + \rho u^2(n)\right]$$

Differentiate with respect to $u(n)$ and equate to zero.

$$2(\frac{E_k B u(n) + F_k y(n)}{C} - \gamma r(n+k))\alpha_0 + 2\rho u(n) = 0$$

where, α_0 is the constant term of $E_k B/C$.

$$(E_k B u(n) + F_k y(n) - \gamma C r(n+k))\alpha_0 + \rho C u(n) = 0$$

Collecting terms to one side,

$$(\alpha_0 E_k B + \rho C) u(n) = (\gamma C r(n+k) - F_k y(n))\alpha_0$$

7. Example: Generalized Minimum Variance Control

$$y(n) = \frac{0.5}{1-0.5z^{-1}}u(n-1) + \frac{1}{1-0.9z^{-1}}\xi(n)$$

$$A = (1-0.5z^{-1})(1-0.9z^{-1}) = 1 - 1.4z^{-1} + 0.45z^{-2}$$

$$B = 0.5(1-0.9z^{-1})$$

$$C = (1-0.5z^{-1})$$

$$k = 1$$

Split C into E_k and F_k:

$$C = E_k A + z^{-k} F_k$$

$$E_1 = 1$$

$$F_1 = 0.9 - 0.45z^{-1}$$

8. Example: Generalized Minimum Variance Control

Recall control law for GMVC. Let $\rho = 1$:

$$u(n) = \frac{\gamma\alpha_0 C}{\alpha_0 E_k B + \rho C} r(n+k) - \frac{\alpha_0 F_k}{\alpha_0 E_k B + \rho C} y(n)$$

$$\alpha_0 = \text{constant value of } \left(\frac{E_k B}{C} = \frac{0.5(1-0.9z^{-1})}{1-0.5z^{-1}}\right) = 0.5$$

$$u(n) = \frac{\gamma 0.5(1-0.5z^{-1})}{0.5\times 0.5(1-0.9z^{-1}) + (1-0.5z^{-1})} r(n+1) - \frac{0.5(0.9-0.45z^{-1})}{0.5\times 0.5(1-0.9z^{-1}) + (1-0.5z^{-1})} y(n)$$

$$= \frac{\gamma 0.4(1-0.5z^{-1})}{1-0.58z^{-1}} r(n+1) - \frac{0.36(1-0.5z^{-1})}{1-0.58z^{-1}} y(n)$$

Think: How do you find γ? At steady state, $y = r$.

9. Generalized Predictive Control: Index to Minimize

Generalized Predictive Control of the following plant:

$$A(z)y(n) = z^{-k}B(z)u(n) + \frac{1}{\Delta}\xi(n)$$

- Want y to follow r (the setpoint).
- Plant has a delay of k:
  - earliest time when the current input $u(n)$ influences the output is $n+k$.
- Thus, we want the plant output to follow a reference trajectory $n+k$ onwards.

Want to minimize the index

$$J = [\hat{y}(n+k) - r(n+k)]^2 + [\hat{y}(n+k+1) - r(n+k+1)]^2 + \cdots$$

where, $\hat{y}$ refers to an estimate of y.

10. GPC: Index to Minimize - Continued

Derived index to minimize:

$$J = [\hat{y}(n+k) - r(n+k)]^2 + [\hat{y}(n+k+1) - r(n+k+1)]^2 + \cdots$$

- We will constrain u also so as to avoid large control effort
- As the noise is assumed to have steps, we may not be able to constrain the absolute value of $u(n)$, but only changes in it.
- To the above index, add the following terms, with $\rho > 0$,

$$\rho(\Delta u(n))^2 + \rho(\Delta u(n+1))^2 + \cdots$$

- Minimize terms up to $n+N$, N large
- Because of control, y will be close to the setpoint $\Rightarrow u$ will become constant after $n+N$ or Δu will become zero. Need to have Δu only up to $n+N$.

11. GPC: Index to Minimize - Continued

Optimization index to minimize:

$$
\begin{aligned}
J_{\text{GPC}} &= [\hat{y}(n+k) - r(n+k)]^2 + \cdots \\
&\quad + [\hat{y}(n+k+N) - r(n+k+N)]^2 \\
&\quad + \rho(\Delta u(n))^2 + \rho(\Delta u(n+1))^2 + \cdots + \rho(\Delta u^2(n+N))^2.
\end{aligned}
$$

Use prediction model, determine $\hat{y}$ and substitute in the above to get an expression for u

12. Prediction Model for GPC

Model of the plant:

$$A(z)y(n) = z^{-k}B(z)u(n) + \xi(n)/\Delta$$

Prediction Model:

$$
\begin{aligned}
\hat{y}(n+j) &= G_j \Delta u(n+j-k) + F_j y(n) \\
G_j &= E_j(z)B(z) \\
1 &= E_j \Delta A + z^{-j} F_j \\
\mathrm{d}E_j &= j - 1 \\
\mathrm{d}F_j &= \mathrm{d}\Delta A - 1 = \mathrm{d}A + 1 - 1 = \mathrm{d}A
\end{aligned}
$$

First term:

$$
\begin{aligned}
G_j \Delta u(n+j-k) &= g_{j,0}\Delta u(n+j-k) + g_{j,1}\Delta u(n+j-k-1) \\
&\quad + \cdots + g_{j,\mathrm{d}G_j}\Delta u(n+j-k-\mathrm{d}G_j)
\end{aligned}
$$

13. Prediction Model for GPC

$$\begin{aligned}
\hat{y}(n+j) &= G_j \Delta u(n+j-k) + F_j y(n) \\
G_j &= E_j(z)B(z) \\
\mathrm{d}G_j &= \mathrm{d}E_j + \mathrm{d}B = j - 1 + \mathrm{d}B, \\
\mathrm{d}E_j &= j-1 \\
\mathrm{d}F_j &= \mathrm{d}\Delta A - 1 = \mathrm{d}A + 1 - 1 = \mathrm{d}A
\end{aligned}$$

The prediction model is,

$$\begin{aligned}
\hat{y}(n+j) = {} & g_{j,0}\Delta u(n+j-k) + g_{j,1}\Delta u(n+j-k-1) \\
& + \cdots + g_{j,\mathrm{d}G_j}\Delta u(n-k+1-\mathrm{d}B) \\
& + f_{j,0}y(n) + f_{j,1}y(n-1) + \cdots + f_{j,\mathrm{d}A}y(n-\mathrm{d}A)
\end{aligned}$$

Split these as
$\hat{y}(n+j) =$ future inputs + past inputs + past outputs

14. $G\Delta u$ Term of GPC Model

Begin with prediction model:

$$\begin{aligned}
\hat{y}(n+j) = {} & g_{j,0}\Delta u(n+j-k) + g_{j,1}\Delta u(n+j-k-1) \\
& + \cdots + g_{j,\mathrm{d}G_j}\Delta u(n-k+1-\mathrm{d}B) + f \text{ terms}
\end{aligned}$$

$$g \text{ terms of } \begin{bmatrix} \hat{y}(n+k) \\ \hat{y}(n+k+1) \\ \vdots \\ \hat{y}(n+k+N) \end{bmatrix}$$

$$= \begin{bmatrix} g_{k,0} & 0 & \cdots & 0 \\ g_{k+1,1} & g_{k+1,0} & \cdots & 0 \\ \vdots & & & \\ g_{k+N,N} & g_{k+N,N-1} & \cdots & g_{k+N,0} \end{bmatrix} \begin{bmatrix} \Delta u(n) \\ \Delta u(n+1) \\ \vdots \\ \Delta u(n+N) \end{bmatrix}$$

$$+ \begin{bmatrix} g_{k,1} & \cdots & g_{k,\mathrm{d}G_k} \\ g_{k+1,2} & \cdots & g_{k+1,\mathrm{d}G_{k+1}} \\ \vdots & & \\ g_{k+N,N+1} & \cdots & g_{k+N,\mathrm{d}G_{k+N}} \end{bmatrix} \begin{bmatrix} \Delta u(n-1) \\ \Delta u(n-2) \\ \vdots \\ \Delta u(n-k+1-\mathrm{d}B) \end{bmatrix}$$

15. F_j Term of GPC Model

Begin with prediction model:

$$\begin{aligned}\hat{y}(n+j) &= g_{j,0}\Delta u(n+j-k) + g_{j,1}\Delta u(n+j-k-1) \\ &+ \cdots + g_{j,\mathrm{d}G_j}\Delta u(n-k+1-\mathrm{d}B) \\ &+ f_{j,0}y(n) + f_{j,1}y(n-1) + \cdots + f_{j,\mathrm{d}A}y(n-\mathrm{d}A)\end{aligned}$$

$$f \text{ terms of } \begin{bmatrix} \hat{y}(n+k) \\ \hat{y}(n+k+1) \\ \vdots \\ \hat{y}(n+k+N) \end{bmatrix}$$

$$= \begin{bmatrix} f_{k,0} & \cdots & f_{k,\mathrm{d}A} \\ f_{k+1,0} & \cdots & f_{k+1,\mathrm{d}A} \\ \vdots \\ f_{k+N,0} & \cdots & f_{k+N,\mathrm{d}A} \end{bmatrix} \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-\mathrm{d}A) \end{bmatrix}$$

Defining vectors $\underline{\hat{y}}$, $\underline{u}$, $\underline{u}_{\mathrm{old}}$ and $\underline{y}_{\mathrm{old}}$ suitably, we obtain

$$\underline{\hat{y}} = G\underline{u} + H_1\underline{u}_{\mathrm{old}} + H_2\underline{y}_{\mathrm{old}}$$

16. GPC Example

$$(1-0.8z^{-1})y(n) = z^{-1}(0.4+0.6z^{-1})u(n) + \frac{1}{\Delta}e(n)$$

$$A = 1-0.8z^{-1}, \quad B = 0.4+0.6z^{-1}, \quad k=1 \quad N=3$$

$$1 = E_j\Delta A + z^{-j}F_j, \quad G_j = E_jB$$

$$E_1 = 1$$

$$E_2 = 1+1.8z^{-1}$$

$$E_3 = 1+1.8z^{-1}+2.44z^{-2}$$

$$E_4 = 1+1.8z^{-1}+2.44z^{-2}+2.9520z^{-3}$$

$$F_1 = 1.8000-0.8000z^{-1}$$

$$F_2 = 2.4400-1.4400z^{-1}$$

$$F_3 = 2.9520-1.9520z^{-1}$$

$$F_4 = 3.3616-2.3616z^{-1}$$

17. GPC Example

$$G_1 = 0.4 + 0.60z^{-1}$$
$$G_2 = 0.4 + 1.32z^{-1} + 1.0800z^{-2}$$
$$G_3 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 1.4640z^{-3}$$
$$G_4 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 2.6448z^{-3} + 1.7712z^{-4}$$

$$G = \begin{bmatrix} g_{k,0} & 0 & \cdots & 0 \\ g_{k+1,1} & g_{k+1,0} & \cdots & 0 \\ \vdots & & & \\ g_{k+N,N} & g_{k+N,N-1} & \cdots & g_{k+N,0} \end{bmatrix}$$
$$= \begin{bmatrix} 0.4000 & 0 & 0 & 0 \\ 1.3200 & 0.4000 & 0 & 0 \\ 2.0560 & 1.3200 & 0.4000 & 0 \\ 2.6448 & 2.0560 & 1.3200 & 0.4000 \end{bmatrix}$$

18. GPC Example

$$G_1 = 0.4 + 0.60z^{-1}$$
$$G_2 = 0.4 + 1.32z^{-1} + 1.0800z^{-2}$$
$$G_3 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 1.4640z^{-3}$$
$$G_4 = 0.4 + 1.32z^{-1} + 2.0560z^{-2} + 2.6448z^{-3} + 1.7712z^{-4}$$

$$H_1 = \begin{bmatrix} g_{k,1} & \cdots & g_{k,\mathrm{d}G_k} \\ g_{k+1,2} & \cdots & g_{k+1,\mathrm{d}G_{k+1}} \\ \vdots & & \\ g_{k+N,N+1} & \cdots & g_{k+N,\mathrm{d}G_{k+N}} \end{bmatrix} = \begin{bmatrix} 0.6000 \\ 1.0800 \\ 1.4640 \\ 1.7712 \end{bmatrix}$$

19. GPC Example

$$F_1 = 1.8000 - 0.8000z^{-1}$$

$$F_2 = 2.4400 - 1.4400z^{-1}$$

$$F_3 = 2.9520 - 1.9520z^{-1}$$

$$F_4 = 3.3616 - 2.3616z^{-1}$$

$$H_2 = \begin{bmatrix} f_{k,0} & \cdots & f_{k,\mathrm{d}A} \\ f_{k+1,0} & \cdots & f_{k+1,\mathrm{d}A} \\ \vdots & & \\ f_{k+N,0} & \cdots & f_{k+N,\mathrm{d}A} \end{bmatrix} = \begin{bmatrix} 1.8000 & -0.8000 \\ 2.4400 & -1.4400 \\ 2.9520 & -1.9520 \\ 3.3616 & -2.3616 \end{bmatrix}$$