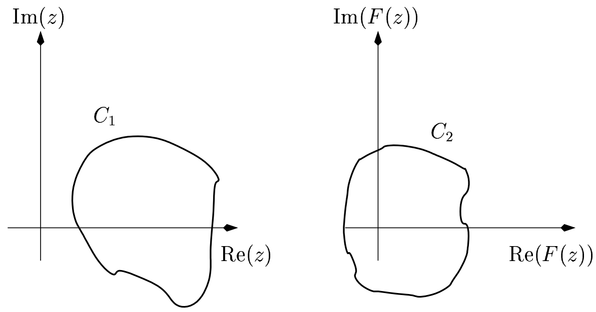
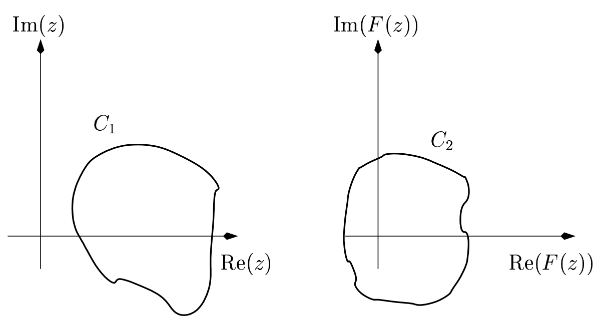
1. Cauchy's Principle for Function $F(z)$

- Draw a closed contour C_1 in Z plane: no zeros/poles of $F(z)$ should lie *on* C_1.



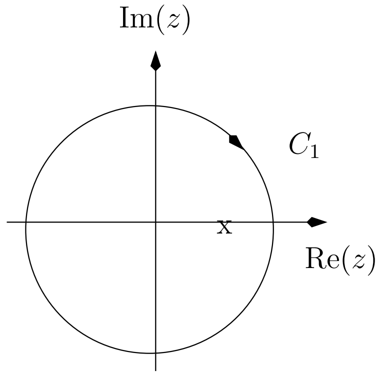
- Let z zeros and p poles of $F(z)$ lie within the closed contour.
- Evaluate $F(z)$ at all points on the curve C_1 in the clockwise direction.

2. Cauchy's Principle for Function $F(z)$



- Plot the evaluated values in another plane, Im$[F(z)]$ *vs.* Re$[F(z)]$.
- The new curve also will be a closed contour, call it C_2.
- Cauchy's Principle: C_2 will encircle origin of F plane N times in clockwise direction. $N = z - p$

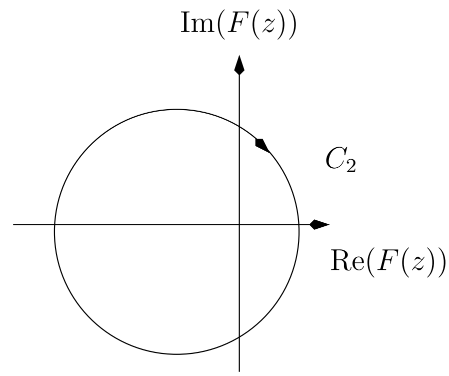
3. Example: Cauchy's Principle



$$F(z) = z - 0.5$$
$$z = e^{j\omega}$$
$$F\left(e^{j\omega}\right) = e^{j\omega} - 0.5$$

- $e^{j\omega}$ is a circle of radius 1, centred at 0.

- Subtract 0.5 from every point.
- It is a circle shifted to the left by 0.5.



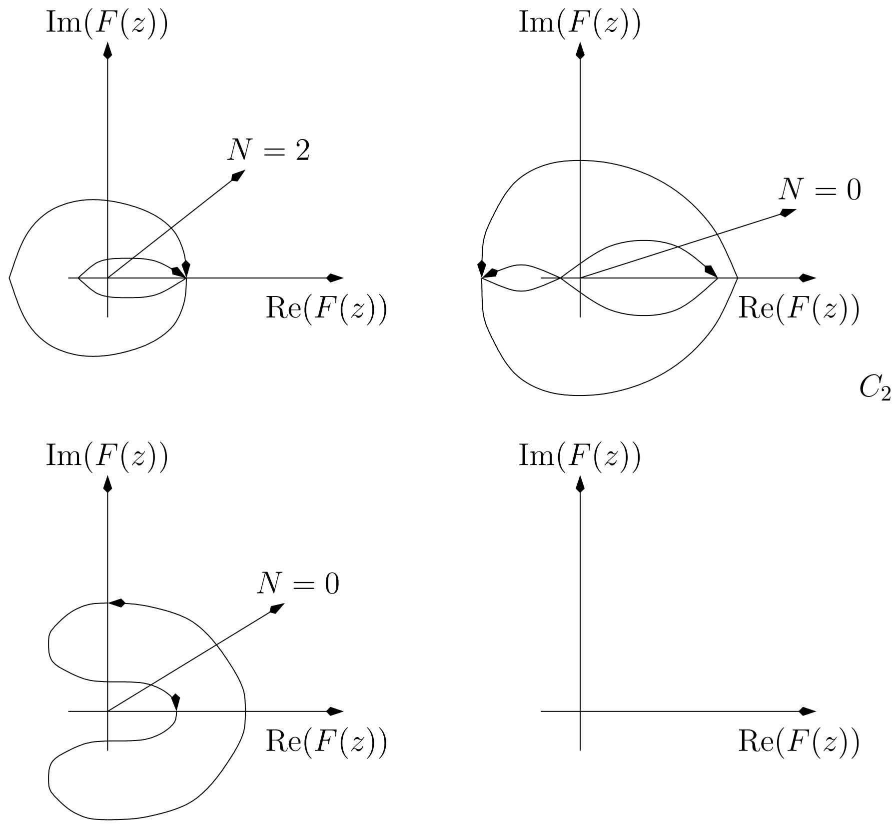
Direction: same as C_1, with one encirclement

4. Example 2: Cauchy's Principle

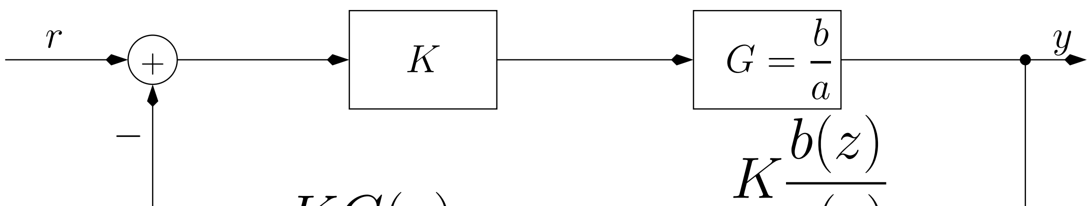
$$F(z) = \frac{1}{z}$$
$$z = e^{j\omega}$$
$$F\left(e^{j\omega}\right) = e^{-j\omega}$$

- It is a unit circle with counter clockwise direction.
- One pole inside C_1, encirclement $= -1$.
- $N = z - p = 0 - 1 = -1$

5. Counting of Encirclements of (0,0)



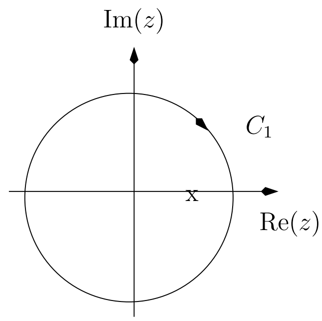
6. Design of Proportional Controller



$$y(n) = \frac{KG(z)}{1+KG(z)} r(n) = \frac{K\dfrac{b(z)}{a(z)}}{1+K\dfrac{b(z)}{a(z)}} r(n)$$

- Zeros of $1 + K\dfrac{b(z)}{a(z)}$ = poles of closed loop system.
- Want them inside unit circle for stability.

7. Encirclement Criterion for Stability



Want zeros of $1 + K\dfrac{b(z)}{a(z)}$ inside unit circle C_1 for stability.

- Let $1 + K\dfrac{b(z)}{a(z)}$ have
  - a total n zeros and n poles.
  - Z zeros outside unit circle, $n - Z$ inside.
  - P poles outside unit circle, $n - P$ inside.
  - P is number of unstable poles
- For stability, $Z = 0$

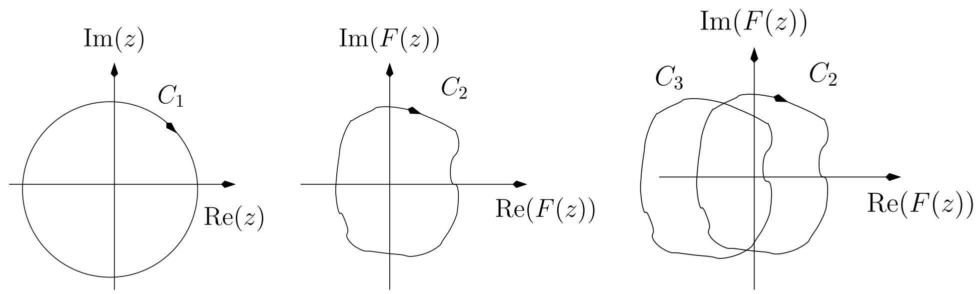
8. Encirclement Criterion for Stability - Continued

- Evaluate $1 + K\dfrac{b(z)}{a(z)}$ along C_1 and plot it: Called C_2
- C_2 encircles origin $N = (n - Z) - (n - P) = P - Z$ times
- Want $Z = 0$ for stability. *i.e.,* $N = P$ for stability

$$\begin{aligned} P &= \text{No. of unstable poles of } 1 + K\frac{b(z)}{a(z)} \\ &= \text{No. of unstable poles of } \frac{a(z) + Kb(z)}{a(z)} \\ &= \text{No. of unstable poles of } \frac{b(z)}{a(z)} \\ &= \text{no. of open loop unstable poles} \end{aligned}$$

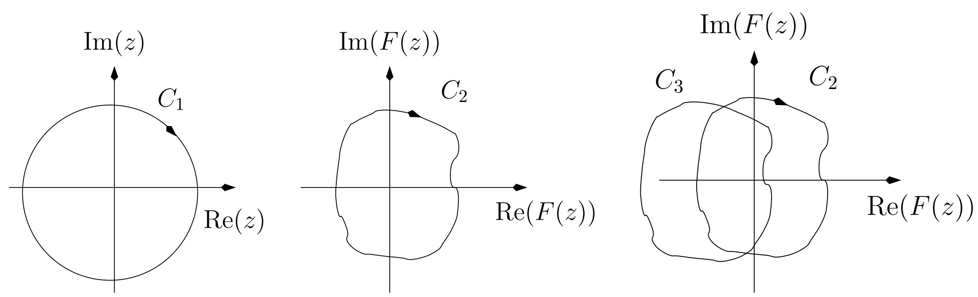
N should be equal to the number of open loop unstable poles

9. Procedure to Calculate K Using Nyquist Plot



- Evaluate $1 + K\dfrac{b(z)}{a(z)}$ along the unit circle (C_1) and plot C_2
- C_2 should encircle origin P times $=$ open loop unstable poles
- K has to be known to do this
- Want to convert this into a design approach to calculate K
- Evaluate $1 + K\dfrac{b(z)}{a(z)} - 1 = K\dfrac{b(z)}{a(z)}$ along C_1, plot, call it C_3

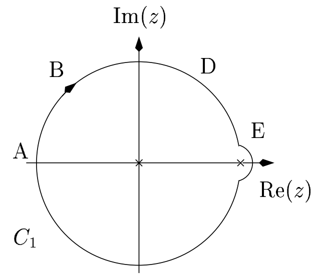
10. Procedure to Calculate K Using Nyquist Plot



- For stability, plot of $Kb(z)/a(z)$, C_3, should encircle the point $(-1, 0)$, P times
- Still need to know K
- Evaluate $\dfrac{b(z)}{a(z)}$ along C_1 and plot. Call it C_4.
- For stability, C_4 should encircle point $(-1/K, 0)$, P times
- C_4 is the Nyquist plot

11. Example of Nyquist Plot to Design Controller

$$G(z) = \frac{b(z)}{a(z)} = \frac{1}{z(z-1)}$$

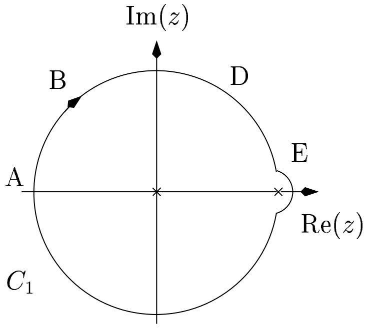


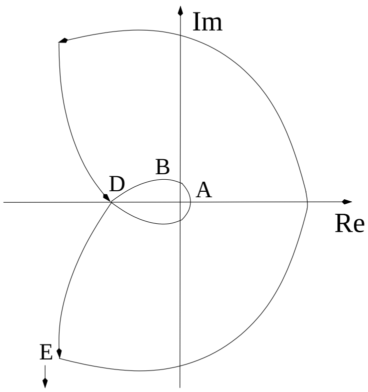
- C_1 should not go through pole/zero
- Indent it with a semicircle of radius $\to 0$
- Number of unstable poles, $P = 0$
- Evaluate $\dfrac{b(z)}{a(z)}$ along main C_1

12. Example of Nyquist Plot to Design Controller

$$\begin{aligned}
G(z) &= \frac{b(z)}{a(z)} = \frac{1}{z(z-1)} \\
G\left(e^{j\omega}\right) &= \frac{1}{e^{j\omega}(e^{j\omega}-1)} \\
&= \frac{1}{e^{j\frac{3}{2}\omega}\left(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}\right)} = -\frac{je^{-j\frac{3}{2}\omega}}{2\sin\frac{\omega}{2}} \\
&= -\frac{j\left(\cos\frac{3}{2}\omega - j\sin\frac{3}{2}\omega\right)}{2\sin\frac{\omega}{2}} \\
&= -\frac{\sin\frac{3}{2}\omega}{2\sin\frac{1}{2}\omega} - j\frac{\cos\frac{3}{2}\omega}{2\sin\frac{1}{2}\omega}
\end{aligned}$$

13. Example of Nyquist Plot - Continued





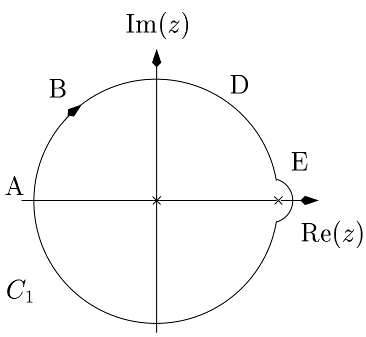
- Transfer function is given by

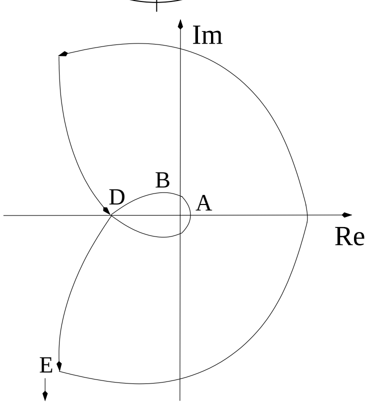
$$G\left(e^{j\omega}\right) = -\frac{\sin \frac{3}{2}\omega}{2\sin\frac{\omega}{2}} - j\frac{\cos\frac{3}{2}\omega}{2\sin\frac{1}{2}\omega}$$

- At point A, $\omega = 180^o$, $G = 0.5$
- At point B, $\omega = 120^o$

$$\begin{aligned} G &= -\frac{\sin\frac{3}{2}120}{2\sin\frac{1}{2}120} - j\frac{\cos\frac{3}{2}120}{2\sin\frac{1}{2}120} \\ &= -\frac{\sin 180}{2\sin 60} - j\frac{\cos 180}{2\sin 60} \\ &= j0.5774 \end{aligned}$$

14. Example of Nyquist Plot - Continued





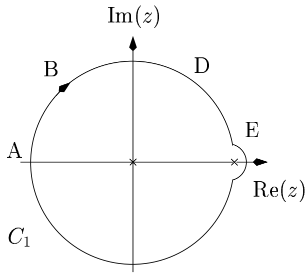
- Transfer function is given by

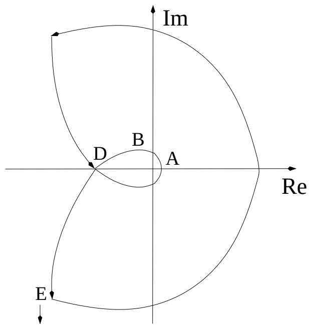
$$G\left(e^{j\omega}\right) = -\frac{\sin \frac{3}{2}\omega}{2\sin\frac{\omega}{2}} - j\frac{\cos\frac{3}{2}\omega}{2\sin\frac{1}{2}\omega}$$

- At point D, $\omega = 60^o$, $G = -\frac{1}{2\frac{1}{2}} - j0 = -1$
- At point E, $\omega \to 0$, $G = -0/0 - j\infty$.
- Use L'Hospital's rule

$$\begin{aligned} G &= -\frac{\frac{3}{2}\cos\frac{3}{2}\omega}{2\frac{1}{2}\cos\frac{1}{2}\omega} - j\infty \\ &= -\frac{3}{2} - j\infty \end{aligned}$$

15. Example of Nyquist Plot - Small Semicircle





$$G(z) = \frac{1}{z(z-1)}$$

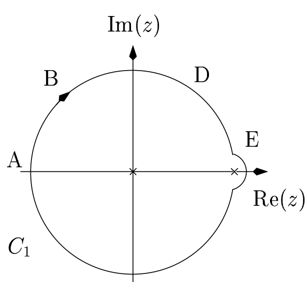
- Indentation around 1
- Semicircle with centre at (1,0), radius $\varepsilon \to 0$
- Because $z = 1 + \varepsilon e^{j\phi}$, ϕ goes from 90^o to 0^o to -90^o

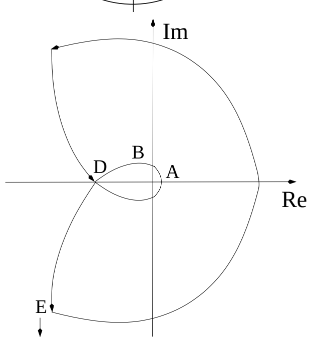
$$G(z) = G\left(1 + \varepsilon e^{j\phi}\right) = \frac{1}{\left(1 + \varepsilon e^{j\phi}\right)\varepsilon e^{j\phi}}$$

$$= \frac{\infty e^{-j\phi}}{1} \text{ as } \varepsilon \to 0$$

- G goes from -90^o through 0^o to 90^o
- Nyquist plot is complete!

16. Example - Controller Design Using Nyquist Plot

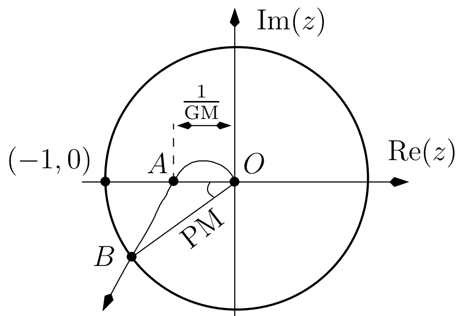




$$G(z) = \frac{1}{z(z-1)}$$

- P = no. of unstable poles of G; $P = 0$
- C_4 should encircle $(-1/K, 0)$ point, P times for stability.
- Should not encircle $(-1/K, 0)$, as $P = 0$
- If $-\dfrac{1}{K}$ is to the left of $(-1, 0)$, stable
- $-1 > -1/K > -\infty$
- $-1 > -1/K$ and $-1/K > -\infty$
- $1 < 1/K$ and $1/K < \infty$
- $K < 1$ and $K > 0$
- $1 > K > 0$

17. Stability Margins

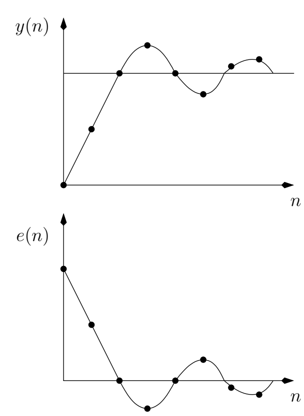


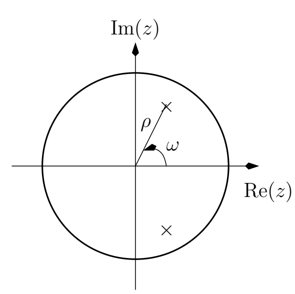
- Nyquist plot (C3) drawn. If passes through $(-1,0)$, unstable.
- Gain margin $=$ 1/OA. If plant transfer function is multiplied by 1/OA, Nyquist plot will go through $(-1,0)$ point and become unstable. Can handle unmodelled gains.
- Phase margin $= \angle AOB$. If rotated clockwise, will go through $(-1,0)$ and become unstable.
- Can handle unmodelled delay. $e^{-j\omega D}$ has phase $= -\omega D$.

18. Specifications - Step Response

Give unit step input in r. Output y should have

1. small rise time
2. small overshoot
3. small settling time
4. small steady state error

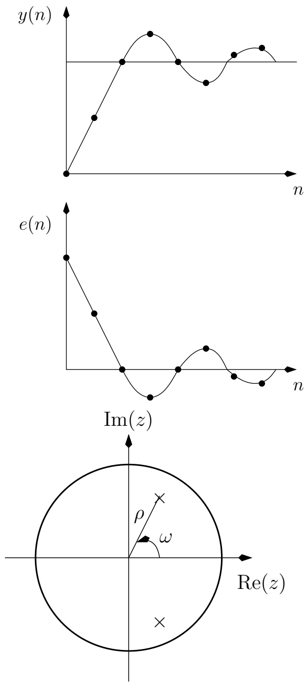




Error $e(n)$ of the following form satisfies the requirements:

$$e(n) = \rho^n \cos \omega n, \quad 0 < \rho < 1$$

19. Specifications - Step Response



1. initial error is one
2. decaying oscillations about zero
3. steady state error is zero

Procedure: User will specify the following:

1. a maximum allowable fall time $< N_r$
2. a maximum allowable undershoot $< \varepsilon$
3. a minimum required decay ratio $< \delta$

- We will develop a method to determine ρ and ω satisfying the above requirements
- Calculate transfer fn. between $e(n)$ - $r(n)$
- Back calculate the controller $G_c(z)$

$$e(n) = \rho^n \cos \omega n$$
$$0 < \rho < 1$$

20. Small Fall Time in Error

$$e(n) = \rho^n \cos \omega n, \quad 0 < \rho < 1$$

Error becomes zero, *i.e.*,

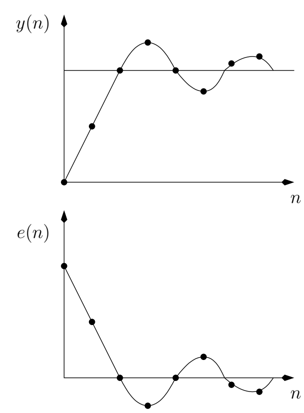
$$e(n) = 0.$$

for the first time when

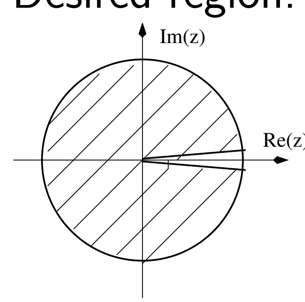
$$\omega n = \frac{\pi}{2}$$
$$\Rightarrow n = \frac{\pi}{2\omega}$$

As want $n < N_r$, some given value, we get

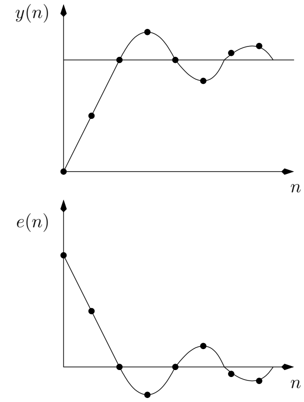
$$\frac{\pi}{2\omega} < N_r$$
$$\Rightarrow \omega > \frac{\pi}{2N_r}$$



Desired region:



21. Small Undershoot



$$e(n) = \rho^n \cos \omega n$$

When does it reach first min.? $de/dn = 0$

$$\rho^n \ln \rho \cos \omega n = \rho^n \omega \sin \omega n$$

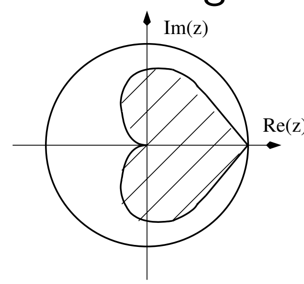
- Look for a simpler expression
- Reaches min. approx. when $\omega n = \pi$

$$e(n)|_{\omega n = \pi} = \rho^n \cos \omega n|_{\omega n=\pi}$$
$$= -\rho^n|_{\omega n=\pi} = -\rho^{\pi/\omega}$$

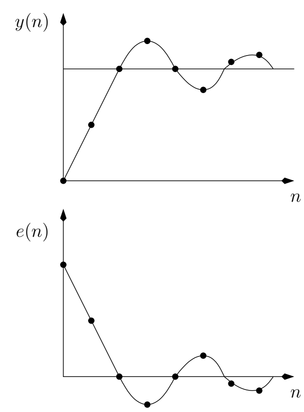
User specified maximum deviation $= \varepsilon$:

$$\rho^{\pi/\omega} < \varepsilon, \quad \rho < \varepsilon^{\omega/\pi}$$

Desired region:



22. Small Decay Ratio



$$e(n) = \rho^n \cos \omega n$$

Ratio of two successive peak/trough to be small

- First undershoot in $e(n)$ occurs at $\omega n \simeq \pi$
- First overshoot occurs at $\omega n \simeq 2\pi$

Want this ratio to be less than user specified δ:

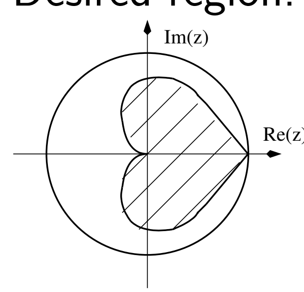
$$\left| \frac{e(n)|_{\omega n=2\pi}}{e(n)|_{\omega n=\pi}} \right| < \delta \Rightarrow \frac{\rho^n|_{\omega n=2\pi}}{\rho^n|_{\omega n=\pi}} < \delta$$

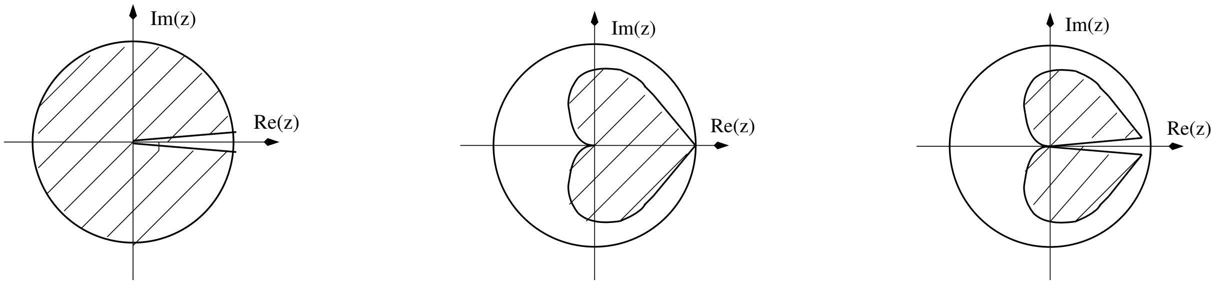
$\delta = 0.5 \simeq 1/4$ decay. $\delta = 0.25 \simeq 1/8$ decay.

$$\frac{\rho^{2\pi/\omega}}{\rho^{\pi/\omega}} < \delta \Rightarrow \rho^{\pi/\omega} < \delta \Rightarrow \rho < \delta^{\omega/\pi}$$

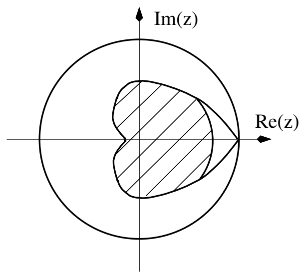
- Small undershoot: $\rho < \varepsilon^{\omega/\pi}$. Usually $\varepsilon < \delta$
- Small undershoot satisfies fast decay

Desired region:





Desired region by the current approach



Obtained by discretization of continuous domain result (Astrom and Wittenmark)