1. Z-transform: Initial value theorem for causal signal

$$u(0) = \lim_{z \to \infty} U(z) \quad \text{if the limit exists}$$

$$U(z) = \sum_{k=-\infty}^{\infty} u(k) z^{-k} = \sum_{k=0}^{\infty} u(k) z^{-k}$$

$$= u(0) + u(1)z^{-1} + u(2)z^{-2} + \cdots$$

$$\lim_{z \to \infty} U(z) = u(0)$$

2. Z-transform: Final value theorem for causal signals

Under the conditions

- $U(z)$ converges for all $|z| > 1$,
- if all the poles of $U(z)(z-1)$ are inside the unit circle,

$$\lim_{k \to \infty} u(k) = \lim_{z \to 1} (z-1)U(z).$$

- Only allowable pole not strictly inside the unit circle is a simple pole at $z = 1$, which is removed in $(z-1)U(z)$
- Allows the important signal of steps to be accommodated
- $U(z)$ is finite for arbitrarily large z implies that $u(k)$ is causal

3. Z-transform: Final value theorem for causal signals

- To prove $\lim_{k\to\infty} u(k) = \lim_{z\to 1}(z-1)U(z)$
- As $u(\infty)$ is bounded we can evaluate the following, which has an extra $u(-1) = 0$:

$$
\begin{aligned}
u(1) &= -u(-1) + u(0) - u(0) + u(1) \\
u(2) &= -u(-1) + u(0) - u(0) + u(1) - u(1) + u(2) \\
\lim_{k\to\infty} u(k) &= \underbrace{-u(-1) + u(0)}_{\Delta u(0)} \underbrace{-u(0) + u(1)}_{\Delta u(1)} - \cdots \\
\lim_{k\to\infty} u(k) &= \Delta u(0) + \Delta u(1) + \Delta u(2) + \cdots = \lim_{z\to 1} \Delta u(0) + \Delta u(1) z^{-1} + \Delta u(2) z^{-2}
\end{aligned}
$$

Define $\Delta u(n) = u(n) - u(n-1)$. Since $\Delta u(k) = 0 \; \forall \; k < 0$,

$$
\begin{aligned}
&= \lim_{z\to 1} \sum_{k=-\infty}^{\infty} \Delta u(k) z^{-k} = \lim_{z\to 1} \sum_{k=-\infty}^{\infty} [u(k) - u(k-1)] z^{-k} \\
&= \lim_{z\to 1} \left[U(z) - z^{-1}U(z)\right] = \lim_{z\to 1} \left(1 - z^{-1}\right) U(z)
\end{aligned}
$$

4. Examples for Final Value Theorem

Using the final value theorem, find the steady state value of $(0.5^n - 0.5)1(n)$ and verify.

$$
\begin{aligned}
(0.5^n - 0.5)1(n) &\leftrightarrow \frac{z}{z-0.5} - \frac{0.5z}{z-1} \quad |z| > 1 \\
\lim_{n\to\infty} LHS &= -0.5 \\
\lim_{z\to 1}(z-1)RHS &= -\lim_{z\to 1} \frac{0.5z}{z-1}(z-1) \\
&= -0.5
\end{aligned}
$$

Is it possible to use the final value theorem on $2^n 1(n)$?

$$
2^n 1(n) \leftrightarrow \frac{z}{z-2} \quad |z| > 2
$$

- Since RHS is valid only for $|z| > 2$, the theorem cannot even be applied.
- In the LHS also, there is a pole outside the unit circle thereby violating the conditions of the theorem.

5. Z-transform of Convolution

If

$$u(n) \leftrightarrow U(z)$$
$$g(n) \leftrightarrow G(z)$$

then,

$$g(n) * u(n) \leftrightarrow G(z)U(z).$$

Recall the motivation slide for Z-transform.

6. Z-transform of Differentiation

If

$$u(n) \leftrightarrow U(z) \text{ with } ROC = R_u \text{ then}$$
$$nu(n) \leftrightarrow -z\frac{dU(z)}{dz} \text{ with } ROC = R_u.$$

Begin with the Z-transform of u:

$$U(z) = \sum_{n=-\infty}^{\infty} u(n)z^{-n}$$
$$\frac{dU(z)}{dz} = \frac{d}{dz}\sum_{n=-\infty}^{\infty} u(n)z^{-n} = -\sum_{n=-\infty}^{\infty} nu(n)z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} nu(n)z^{-n}$$
$$-z\frac{dU(z)}{dz} = \sum_{n=-\infty}^{\infty} nu(n)z^{-n} \qquad \text{or} \qquad nu(n) \leftrightarrow -\frac{zdU(z)}{dz}$$

7. Important Result from Differentiation

Problem 4.9 in Text: Consider

$$1(n)a^n \leftrightarrow \frac{z}{z-a} = \sum_{n=0}^{\infty} a^n z^{-n}, \quad |az^{-1}| < 1$$

Differentiating w.r.t. a, can show that

$$\frac{z}{(z-a)^2} = \sum_{n=0}^{\infty} n a^{n-1} z^{-n}$$

and hence that

$$n a^{n-1} 1(n) \leftrightarrow \frac{z}{(z-a)^2}$$

Differentiating once again,

$$n(n-1)a^{n-2}1(n) \leftrightarrow \frac{2z}{(z-a)^3}$$

Example

$$n^2 1(n) = [n(n-1) + n]\, 1(n)$$

Notice that $a = 1$ now.

Taking Z-transform,

$$\begin{aligned} &\leftrightarrow \frac{2z}{(z-1)^3} + \frac{z}{(z-1)^2} \\ &= \frac{z^2 + z}{(z-1)^3} \end{aligned}$$

8. Z-Transform of Folded or Time Reversed Functions

If Z-transform of $u(n)$ is $U(z)$, the Z-transform of $u(-n)$ is $U(z^{-1})$.

Proof:

$$\begin{aligned} Z[u(-n)] &= \sum_{n=-\infty}^{\infty} u(-n) z^{-n} \\ &= \sum_{m=-\infty}^{\infty} u(m) z^{m}, \text{ where, } m = -n \\ &= \sum_{m=-\infty}^{\infty} u(m) (z^{-1})^{-m} \\ &= U(z^{-1}). \end{aligned}$$

9. Z-transform of Discrete State Space Systems

$$x(n+1) = Ax(n) + Bu(n) \quad x(0) = x_0 \tag{1}$$
$$y(n) = Cx(n) + Du(n) \tag{2}$$

Eq. 1 is invalid for $n = -1$:

$$x(0) = Ax(-1) + Bu(-1) = 0 \neq x_0$$

$n < 0$ property not explicitly stated. But if we write it as

$$x(n+1) = Ax(n) + Bu(n) + \delta(n+1)x_0 \tag{3}$$

and assume initial rest, all variables are zero until $n = 0$, problem is solved:

- It satisfies the condition for all n: (1) $n < 0$ (2) $n = 0$ (3) $n > 0$
- Meaning: All variables are zero to start with. Somehow $x = x_0$ at $n = 0$.
- Makes the model well defined for all n. Can take Z-transform as well
- Using one sided Z-transform leaves the problem statement vague

10. Z-transform of Discrete State Space Systems - Ctd.

Z-transform of $x(n+1) = Ax(n) + Bu(n) + \delta(n+1)x_0$ gives

$$\begin{aligned} zX(z) &= AX(z) + BU(z) + x_0 z \\ (zI - A)X(z) &= BU(z) + x_0 z \\ X(z) &= (zI - A)^{-1}BU(z) + z(zI - A)^{-1}x_0 \end{aligned}$$

Z-transform of $y(n) = Cx(n) + Du(n)$ is

$$\begin{aligned} Y(z) &= CX(z) + DU(z) \\ &= C(zI - A)^{-1}BU(z) + DU(z) + C(zI - A)^{-1}zx(0) \\ &= [C(zI - A)^{-1}B + D]U(z) + C(zI - A)^{-1}zx(0) \\ &\triangleq \underbrace{Y_u(z)}_{\text{Z-transform of } y_u} + \underbrace{Y_x(z)}_{\text{Z-transform of } y_x} \\ &\triangleq G_u(z)U(z) + G_x(z)x_0 \end{aligned}$$

$$\begin{aligned} C(zI - A)^{-1}B &= G_u(z) \leftrightarrow CA^{n-1}B \\ (zI - A)^{-1}z &\leftrightarrow A^n \end{aligned}$$

11. Recall: Recursive Solution to Discrete State Equation

$$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k) + Du(k),\ \{u(k)\} = \{u(0), u(1), u(2), \ldots\}$$

$$x(1) = Ax(0) + Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = A[Ax(0) + Bu(0)] + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$

$$x(3) = Ax(2) + Bu(2) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)$$

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-(i+1)} Bu(i), \qquad A^o = I$$

$$y(k) = \underbrace{CA^k x(0)}_{\text{state response } y_x} + \underbrace{\sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + Du(k)}_{\text{input response } y_u}$$

In input-output setting, we get

$$y(k) = y_x + y_u = y_x + \sum_{i=0}^{k} u(i)g(k-i) = y_x + \sum_{i=0}^{k-1} u(i)g(k-i) + u(k)g(0)$$

Comparing terms, we get,

$$g(k) = CA^{k-1}B,\ k > 0, \quad g(0) = D$$

Usually, however, D and hence $g(0)$, are zero.

12. Finding Transfer Function - an Example

Find the transfer function of

$$A = \begin{bmatrix} 1 & 0 \\ 0.19801 & 0.9802 \end{bmatrix}, \quad B = \begin{bmatrix} 0.02 \\ 0.001987 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0, G(z) = c(zI - A)^{-1}B$$

$$G = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ -0.19801 & z-0.9802 \end{bmatrix}^{-1} \begin{bmatrix} 0.02 \\ 0.001987 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}}{(z-1)(z-0.9802)} \begin{bmatrix} z-0.9802 & 0 \\ 0.19801 & z-1 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.001987 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 0.19801 & z-1 \end{bmatrix}}{(z-1)(z-0.9802)} \begin{bmatrix} 0.02 \\ 0.001987 \end{bmatrix}$$

$$= \frac{0.001987z + 0.0019732}{(z-1)(z-0.9802)} = 0.001987 \frac{z + 0.9931}{(z-1)(z-0.9802)}$$

```
F = [0 0;1 -0.1]; G = [0.1; 0];
C = [0 1]; D = 0; Ts = 0.2;
sys = ss(F,G,C,D);
sysd = c2d(sys,Ts,'zoh');
H = tf(sysd)
```

13. Inverse Z-transform - Partial Fraction

Find the inverse Z-transform of

$$G(z) = \frac{2z^2 + 2z}{z^2 + 2z - 3}.$$

$$\frac{G(z)}{z} = \frac{2z+2}{(z+3)(z-1)} \quad |z| > 3$$

$$= \frac{A}{z+3} + \frac{B}{z-1} \quad |z| > 3$$

Multiply throughout by $z + 3$ and let $z = -3$ to get

$$A = \left.\frac{2z+2}{z-1}\right|_{z=-3} = \frac{-4}{-4} = 1.$$

Multiply throughout by $z - 1$ and let $z = 1$ to get

$$B = \frac{4}{4} = 1$$

$$\frac{G(z)}{z} = \frac{1}{z+3} + \frac{1}{z-1} \quad |z| > 3$$

$$G(z) = \frac{z}{z+3} + \frac{z}{z-1} \quad |z| > 3$$

$$\leftrightarrow (-3)^n 1(n) + 1(n)$$

14. Partial Fraction - Repeated Poles

$$G(z) = \frac{N(z)}{(z-\alpha)^p D_1(z)} \quad \alpha \text{ not a root of } N(z) \text{ and } D_1(z)$$

$$G(z) = \frac{A_1}{z-\alpha} + \frac{A_2}{(z-\alpha)^2} + \cdots + \frac{A_p}{(z-\alpha)^p} + G(z),$$

$G(z)$ has poles corresponding to those of $D_1(z)$. Multiply by $(z-\alpha)^p$,

$$(z-\alpha)^p G(z) = A_1(z-\alpha)^{p-1} + A_2(z-\alpha)^{p-2} + \cdots$$
$$+ A_{p-1}(z-\alpha) + A_p + G(z)(z-\alpha)^p$$

Substituting $z = \alpha$, $\quad A_p = (z-\alpha)^p G(z)|_{z=\alpha}$.

Differentiate and let $z = \alpha$: $\quad A_{p-1} = \frac{d}{dz}(z-\alpha)^p G(z)|_{z=\alpha}$

Continuing, $\quad A_1 = \frac{1}{(p-1)!}\frac{d^{p-1}}{dz^{p-1}}(z-\alpha)^p G(z)|_{z=\alpha}$

15. Partial Fraction for Repeated Poles - an Example

$$G(z) = \frac{11z^2 - 15z + 6}{(z-2)(z-1)^2} = \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2} + \frac{B}{z-2}$$

Multiply by $z-2$, let $z=2$, to get $B=20$. Multiply by $(z-1)^2$,

$$\frac{11z^2 - 15z + 6}{z-2} = A_1(z-1) + A_2 + B\frac{(z-1)^2}{z-2}$$

With $z=1$, get $A_2 = -2$. Differentiating with respect to z and with $z=1$,

$$\begin{aligned}
A_1 &= \left.\frac{(z-2)(22z-15) - (11z^2 - 15z + 6)}{(z-2)^2}\right|_{z=1} = -9 \\
G(z) &= -\frac{9}{z-1} - \frac{2}{(z-1)^2} + \frac{20}{z-2}. \\
zG(z) &= -\frac{9z}{z-1} - \frac{2z}{(z-1)^2} + \frac{20z}{z-2}, \\
&\leftrightarrow (-9 - 2n + 20.2^n)1(n), \\
G(z) &\leftrightarrow (-9 - 2(n-1) + 20.2^{n-1})1(n-1)
\end{aligned}$$

16. Partial Fraction: Num. Degree = Den. Degree

If numerator degree = denominator degree, divide by denominator, and do a partial fraction expansion:

$$\begin{aligned}
G(z) &= \frac{(z^3 - z^2 + 3z - 1)}{(z-1)(z^2 - z + 1)} \\
&= \left[1 + \frac{z(z+1)}{(z-1)(z^2 - z + 1)}\right] \\
&\triangleq (1 + G'(z))
\end{aligned}$$

As $G'(z)$ has a zero at the origin, its partial fraction expansion is carried out as follows:

$$\begin{aligned}
\frac{G'(z)}{z} &= \frac{z+1}{(z-1)(z^2 - z + 1)} \\
&= \frac{z+1}{(z-1)(z - e^{j\pi/3})(z - e^{-j\pi/3})} \\
&= \frac{2}{z-1} - \frac{1}{z - e^{j\pi/3}} - \frac{1}{z - e^{-j\pi/3}}. \\
G'(z) &= \frac{2z}{z-1} - \frac{z}{z - e^{j\pi/3}} - \frac{z}{z - e^{-j\pi/3}}. \\
&\leftrightarrow 2 - e^{j\pi k/3} - e^{-j\pi k/3} \\
&= \left(2 - 2\cos\frac{\pi}{3}k\right) 1(k).
\end{aligned}$$

Finally, from this, we get the inverse of the given transform as

$$g(k) = \delta(k) + \left(2 - 2\cos\frac{\pi}{3}k\right) 1(k).$$

17. Partial Fraction - Powers of z^{-1}

$$G(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} \quad \left(|z| > \frac{1}{3}\right) = \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}}$$

Multiply both sides by $1 - \frac{1}{4}z^{-1}$ and let $z = \frac{1}{4}$ to get

$$A = \left.\frac{3 - \frac{5}{6}z^{-1}}{1 - \frac{1}{3}z^{-1}}\right|_{z=\frac{1}{4}} = 1$$

Multiply both sides by $1 - \frac{1}{3}z^{-1}$ and let $z = \frac{1}{3}$ to get

$$B = \left.\frac{3 - \frac{5}{6}z^{-1}}{1 - \frac{1}{4}z^{-1}}\right|_{z=\frac{1}{3}} = 2.$$

Substituting in the above, we get,

$$G(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}} \leftrightarrow \left[\left(\frac{1}{4}\right)^n + 2\left(\frac{1}{3}\right)^n\right] 1(n)$$

18. Power Series Method to Invert Z-Transform

Write numerator and denominator in powers of z^{-1} and divide.

$$G(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Apply method of long division:

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & 1 \quad -az^{-1} \\ \hline & \quad az^{-1} \\ & \quad az^{-1} \quad -a^2z^{-2} \\ \hline & \qquad\qquad a^2z^{-2} \end{array}$$

To summarize,

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

As coefficients of positive powers of z are zero,

$$\begin{aligned} g(n) &= 0, \qquad n < 0 \\ g(0) &= 1 \\ g(1) &= a \\ g(2) &= a^2 \end{aligned}$$

Generalizing,

$$g(n) = a^n 1(n).$$

19. Controller Implementation: Realization, Inversion of $G(z)$

Implementation of controller through inversion: known as realization.

$$G(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots} \triangleq \frac{B(z)}{A(z)}$$

If the input to this system is $U(z)$ and the corresponding output is $Y(z)$,

$$Y(z) = G(z)U(z) = \frac{B(z)}{A(z)}U(z)$$

Cross multiplying,

$$A(z)Y(z) = B(z)U(z)$$

Using the expression for B and A, the above equation becomes,

$$\begin{aligned} Y(z) + a_1 z^{-1}Y(z) + a_2 z^{-2}Y(z) \\ + a_3 z^{-3}Y(z) + \cdots \\ = b_0 U(z) + b_1 z^{-1}U(z) \\ + b_2 z^{-2}U(z) + b_3 z^{-3}U(z) + \cdots \end{aligned}$$

$$\begin{aligned} Y(z) = & -a_1 z^{-1}Y(z) - a_2 z^{-2}Y(z) \\ & - a_3 z^{-3}Y(z) - \cdots \\ & + b_0 U(z) + b_1 z^{-1}U(z) \\ & + b_2 z^{-2}U(z) + b_3 z^{-3}U(z) + \cdots \end{aligned}$$

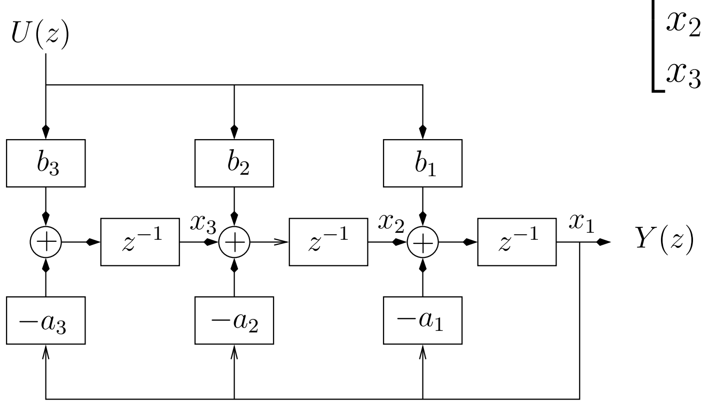
Using the Shifting Theorem,

$$\begin{aligned} y(n) = & -a_1 y(n-1) - a_2 y(n-2) \\ & - a_3 y(n-3) + \cdots \\ & + b_0 u(n) + b_1 u(n-1) \\ & + b_2 u(n-2) + b_3 u(n-3) + \cdots \end{aligned}$$

20. State Space Realization

$$Y(z) = G(z)U(z) = \frac{B(z)}{A(z)}U(z)$$

$$\begin{aligned} Y(z) = & -a_1 z^{-1}Y(z) - a_2 z^{-2}Y(z) \\ & - a_3 z^{-3}Y(z) - \cdots \\ & + b_0 U(z) + b_1 z^{-1}U(z) \\ & + b_2 z^{-2}U(z) + b_3 z^{-3}U(z) + \cdots \end{aligned}$$



$$\begin{aligned} y(k) &= x_1(k) \\ zx_1(k) &= x_2(k) + b_1 u(k) - a_1 x_1(k) \\ zx_2(k) &= x_3(k) + b_2 u(k) - a_2 x_1(k) \\ zx_3(k) &= b_3 u(k) - a_3 x_1(k) \end{aligned}$$

In the form of state space equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$