1. External (BIBO) Stability of LTI Systems

If every Bounded Input produces Bounded Output,

- system is externally stable
- equivalently, system is BIBO stable

$$\sum_{n=-\infty}^{\infty} |g(n)| < \infty \Leftrightarrow \text{BIBO Stability}$$

- Don't care about what unbounded input does...

2. Condition for BIBO Stability

$$\sum_{n=-\infty}^{\infty} |g(n)| < \infty \Leftrightarrow \text{BIBO Stability}$$

$$y(n) = \sum_{k=-\infty}^{\infty} g(k)u(n-k)$$

- Suppose $\sum_{n=-\infty}^{\infty} |g(n)| < \infty$.
- Bounded u with $|u(n)| < M \quad \forall n$
- Evaluate one term:

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |g(k)||u(n-k)|$$
$$\leq M \sum_{k=-\infty}^{\infty} |g(k)|$$

As M is bounded, so is $y(n)$. This is true for all n.

Is it possible for system to be BIBO stable and $\sum |g(i)| \to \infty$?

Produce the following signal:

$$u(k) = \begin{cases} \frac{g(-k)}{|g(-k)|} & g(-k) \neq 0, \\ 0 & g(-k) = 0. \end{cases}$$

$$\{u(n)\} = \{\ldots, 1, -1, -1, 1, -1, \ldots\}$$

$$y(0) = \sum_{k=-\infty}^{\infty} u(k)g(n-k)|_{n=0}$$
$$= \sum_{k=-\infty}^{\infty} u(k)g(-k)$$
$$= \sum_{k=-\infty}^{\infty} \frac{[g(-k)]^2}{|g(-k)|} = \sum_{k=-\infty}^{\infty} |g(-k)|$$

- So, not BIBO stable

3. Z-transform - Motivation

$$y(n) = \sum_{k=-\infty}^{\infty} u(k)g(n-k)$$

$$\{u\} = \{u(0), u(1), u(2)\}$$
$$\{g\} = \{g(0), g(1), g(2)\}$$

Let these be zero at all other times.

$$y(0) = u(0)g(0)$$
$$y(1) = u(0)g(1) + u(1)g(0)$$
$$y(2) = u(0)g(2) + u(1)g(1) + u(2)g(0)$$
$$y(3) = u(1)g(2) + u(2)g(1)$$

Also carryout multiplication:

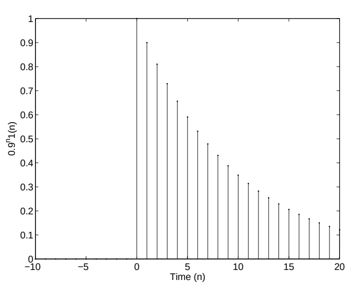
$$(u(0) + u(1)z^{-1} + u(2)z^{-2})\times$$
$$(g(0) + g(1)z^{-1} + g(2)z^{-2}) =$$

$$u(0)g(0)+$$
$$(u(0)g(1) + u(1)g(0))z^{-1}+$$
$$(u(0)g(2) + u(1)g(1) + u(2)g(0))z^{-2}+$$
$$(u(1)g(2) + u(2)g(1))z^{-3}+$$

- Think of z as a position marker - coefficient of z^{-i} occurs at i^{th} instant
- $u(0) + u(1)z^{-1} + u(2)z^{-2}$ - a way of representing a sequence with three terms: $\{u(0), u(1), u(2)\}$
- The z polynomial can often be represented by a compact expression even if the original sequence has infinitely many nonzero terms

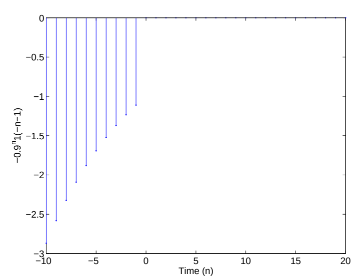
4. Two Sequences to Show Importance of Convergence

$$u_1(n) = a^n 1(n)$$




```
a = 0.9;
n = -10:20;
y = zeros(size(n));
for i = 1:length(n)
    if n(i)>=0,
        y(i) = a^n(i);
    end
end
A = axes('FontSize',14);
set(get(A,'Xlabel'),'FontSize',14);
o = stem(n,y);
set(o(1),'Marker','.');
xlabel('Time (n)');
ylabel('0.9^n1(n)');
```

$$u_2(n) = -a^n 1(-n-1)$$




```
a = 0.9;
n = -10:20;
y = zeros(size(n));
for i = 1:length(n)
    if n(i)<=-1,
        y(i) = -(a^n(i));
    end
end
A = axes('FontSize',14);
set(get(A,'Xlabel'),'FontSize',14);
o = stem(n,y);
set(o(1),'Marker','.');
xlabel('Time (n)');
ylabel('-0.9^n1(-n-1)');
```

5. Z-transform - Convergence Condition for Uniqueness

$$u_1(n) = a^n 1(n)$$

Take Z-transform:

$$U_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

If the sum exists,

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

$$u_2(n) = -a^n 1(-n-1)$$

$$U_2(z) = -\sum_{n=-\infty}^{\infty} a^n 1(-n-1) z^{-n} = -\sum_{n=-1}^{-\infty} a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

If the sum exists,

$$= 1 - \frac{1}{1 - a^{-1}z} = \frac{z}{z-a}$$

Any difference? Clue: If the sum exists. Convergence Condition:

$$az^{-1} < 1$$
$$a < z$$

$$a^{-1}z < 1$$
$$z < a$$

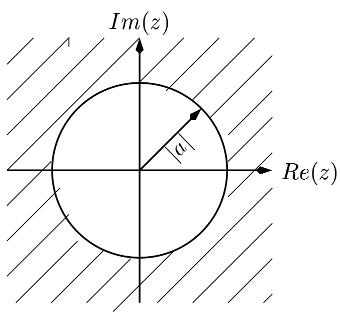
6. Z-transform - Convergence Condition for Uniqueness

$$u_1(n) = a^n 1(n)$$
$$U_1(z) = \frac{z}{z-a}$$
$$az^{-1} < 1$$

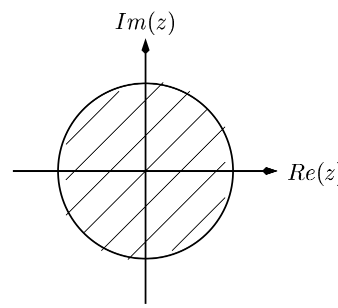
$$u_2(n) = -a^n 1(-n-1)$$
$$U_2(z) = \frac{z}{z-a}$$
$$a^{-1}z < 1$$

- May want a complex, would want z complex
- So make the convergence absolute

$$|az^{-1}| < 1$$
$$|a| < |z|$$



$$|a^{-1}z| < 1$$
$$|z| < |a|$$



7. Z-transform Definition

Z-transform of a sequence $\{u(n)\}$, denoted by $U(z)$, calculated using:

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

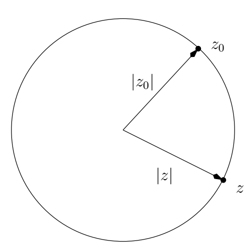
where z is such that

$$U(z) = \sum_{n=-\infty}^{\infty} |u(n) z^{-n}| < \infty$$

- Stronger condition of absolute convergence allows u and z to be complex.
- The main advantage in taking Z-transform is that cumbersome convolution calculations can be done easier.
- Suitable for studying stability, causality and oscillatory behaviour of even infinitely long signals. Will use this fact while designing controllers.
- There is an added burden of converting the polynomial to a closed form expression and inverting this procedure to arrive at the resultant polynomial. It is possible to develop techniques to simplify these operations.

8. Region of Convergence (ROC) - Property 1

ROC consists of a ring centred around the origin



$$\begin{aligned} U(z) &= \sum u(n) z^{-n} \\ z_0 \in ROC &\Rightarrow \sum |u(n) z_0^{-n}| < \infty \\ \infty &> \sum |u(n) z_0^{-n}| = \sum |u(n)||z_0^{-n}| = \sum |u(n)||z_0|^{-n} \overset{|z|=|z_0|}{=} \sum |u(n)||z|^{-n} \\ &= \sum |u(n) z^{-n}| \end{aligned}$$

Thus, arbitrary z also belongs to ROC

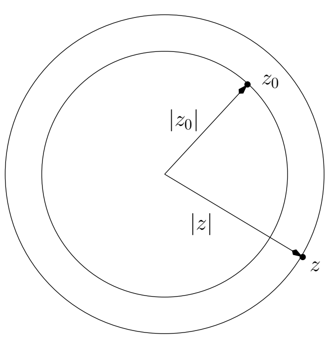
9. ROC Property 2 - Poles are Outside ROC

ROC does not have any pole

At a pole, transfer function is infinite, does not converge

10. ROC - 3: Larger Circles are in ROC for Causal Signals

For sequence $\{u(n)\}$, with $u(n) = 0$ for $n < 0$, if $z_0 \in ROC$, all z such that $|z| > |z_0|$ will belong to ROC



$$U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$z_0 \in ROC \Rightarrow \sum_{n=0}^{\infty} |u(n) z_0^{-n}| < \infty$$

$$\begin{aligned}
\infty > \sum_{n=0}^{\infty} |u(n) z_0^{-n}| \\
&= \sum_{n=0}^{\infty} |u(n)||z_0^{-n}| \\
&= \sum_{n=0}^{\infty} |u(n)||z_0|^{-n} = \sum_{n=0}^{\infty} \frac{|u(n)|}{|z_0|^n} \\
&> \sum_{n=0}^{\infty} \frac{|u(n)|}{|z|^n}, \quad \because |z| > |z_0| \\
&= \sum_{n=0}^{\infty} |u(n) z^{-n}|
\end{aligned}$$

Thus, arbitrary z also belongs to ROC

11. ROC - 4: Poles of Causal, Stable Systems in Unit Circle

Causal and stable system

- Impulse response $g(n)$
- Z-transform of $g(n)$, namely $G(z)$, will have poles inside unit circle

As $g(k)$ is causal and stable,

$$\sum_{k=0}^{\infty} |g(k)| < \infty$$

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

$$\infty > \sum_{k=0}^{\infty} |g(k)| = \sum_{k=0}^{\infty} \left| g(k) z^{-k} \right|_{z=1}$$

- Thus, there is absolute convergence at $|z| = 1$.
- Equivalently, the unit circle belongs to the region of convergence.
- As it is a causal sequence, by Property 3, all points outside the unit circle also belong to the ROC
- From property 2, the poles cannot lie in ROC - they have to lie within the unit circle

12. ROC - 5: Z-transform of Causal Signal is Proper

$u(k)$ is a causal sequence

- $U(z) = \frac{N(z)}{D(z)}$ with
  - $N(z)$ is a polynomial of degree n
  - $D(Z)$ is a polynomial of degree m
- $n \leq m$.

- Since $u(k)$ is causal, by property 3, $z = \infty$ is in ROC
- If $n > m$, $U(z)$ will diverge at ∞
- Thus $n \leq m$.

13. Z-transform Theorems - Linearity

$$Z\left[\alpha\{u_1(n)\} + \beta\{u_2(n)\}\right] = \alpha U_1(z) \; + \beta U_2(z)$$

where,

$$\begin{aligned} u_1(n) &\leftrightarrow U_1(z) \\ u_2(n) &\leftrightarrow U_2(z). \end{aligned}$$

Here α and β are arbitrary scalars.

$$\begin{aligned} LHS &= \sum_{n=-\infty}^{\infty} \left[\alpha u_1(n) + \beta u_2(n)\right] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \alpha u_1(n) z^{-n} \; + \sum_{n=-\infty}^{\infty} \beta u_2(n) z^{-n} \\ &= \alpha U_1(z) + \beta U_2(z) = RHS \end{aligned}$$

14. Example 1 - Linearity

Find the Z-transform of

$$\begin{aligned} u_1(n) &= 2\delta(n) - 3\delta(n-2) \\ &\quad + 4\delta(n-5) \\ U_1(z) &= 2\sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \\ &\quad - 3\sum_{n=-\infty}^{\infty} \delta(n-2) z^{-n} \\ &\quad + 4\sum_{n=-\infty}^{\infty} \delta(n-5) z^{-n} \\ &= 2 - 3z^{-2} + 4z^{-5} \end{aligned}$$

$\forall z^{-1}$ finite

$$= \frac{2z^5 - 3z^3 + 4}{z^5} \quad |z| > 0.$$

Find the Z-transform of

$$\{u_2(n)\} = \left[2 + 4(-3)^n\right]\{1(n)\}$$

$$\begin{aligned} U_2(z) &= \sum_{n=0}^{\infty} \left[2 + 4(-3)^n\right] z^{-n} \\ &= \frac{2z}{z-1} + \frac{4z}{z+3}, \quad |z| > 1 \cap |z| > 3 \\ &= \frac{6z^2 + 2z}{(z-1)(z+3)}, \quad |z| > 3 \end{aligned}$$

15. Example 2 - Linearity

Find the Z-transform of $\cos wn\, 1(n)$ and $\sin wn\, 1(n)$.

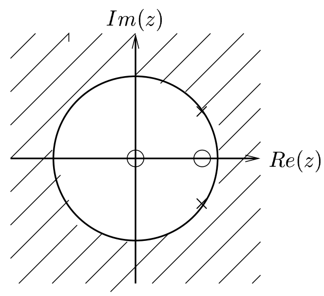
$$\cos\omega n + j\sin\omega n = e^{j\omega n}$$
$$Z\left[\cos\omega n + j\sin\omega n\right] = Z\left\{e^{j\omega n}\right\}$$

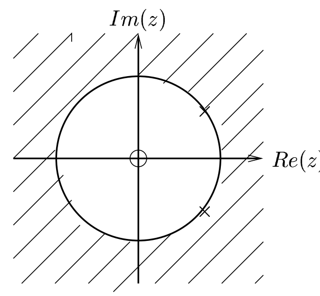
$$
\begin{aligned}
Z\left[e^{j\omega n}\,1(n)\right] &= \sum_{n=-\infty}^{\infty} e^{j\omega n}\, z^{-n}\; 1(n) \\
&= \sum_{n=0}^{\infty} e^{j\omega n}\, z^{-n} \\
&= \frac{z}{z-e^{j\omega}},\quad |z| > |e^{j\omega}| = 1 \\
&= \frac{z(z-e^{-j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})} \\
&= \frac{z(z-\cos\omega + j\sin\omega)}{(z-e^{j\omega})(z-e^{-j\omega})} \\
&= \frac{z(z-\cos\omega)}{(z-e^{j\omega})(z-e^{-j\omega})} + j\frac{z\sin\omega}{(z-e^{j\omega})(z-e^{-j\omega})}
\end{aligned}
$$

Comparing real and imaginary parts,

$$
\begin{aligned}
Z\left[\cos\omega n\right] &= Re\left[Z\left\{e^{j\omega n}\right\}\right] \\
Z\left[\sin\omega n\right] &= Im\left[Z\left\{e^{j\omega n}\right\}\right].
\end{aligned}
$$

$$(\cos\omega n)1(n) \leftrightarrow \frac{z(z-\cos\omega)}{(z-e^{j\omega})(z-e^{-j\omega})}$$
$$(\sin\omega n)1(n) \leftrightarrow \frac{z\sin\omega}{(z-e^{j\omega})(z-e^{-j\omega})}$$





16. Z-transform - Shifting

$$
\begin{aligned}
Z\left[u(k+d)\right] &= z^d U(z) \\
Z\left[u(k+d)\right] &= \sum_{k=-\infty}^{\infty} u(k+d) z^{-k} \\
&= z^d \sum_{k=-\infty}^{\infty} u(k+d) z^{-(k+d)} \\
&= z^d U(z)
\end{aligned}
$$

Example:

If

$$\{u(n)\} \leftrightarrow U(z),$$

then

$$
\begin{aligned}
\{u(n+3)\} &\leftrightarrow z^3 U(z) \\
\{u(n-2)\} &\leftrightarrow z^{-2} U(z)
\end{aligned}
$$