1. Linearity of a Function

A function $f(x)$ is defined linear if

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$$

where α and β are scalars.
Example of a linear function:

$$f(x) = 2x$$

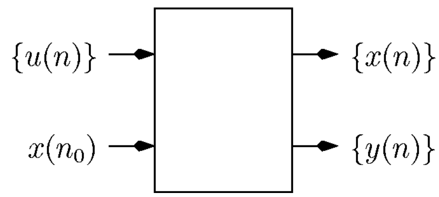
A nonlinear function:

$$f(x) = \sin x$$

What about

$$f(x) = 2x + 3?$$

System seen as consisting of two functions: future states and outputs:



1. Inputs:

   $\{u(n)\}$: input sequence
   $x(n_0)$: initial state

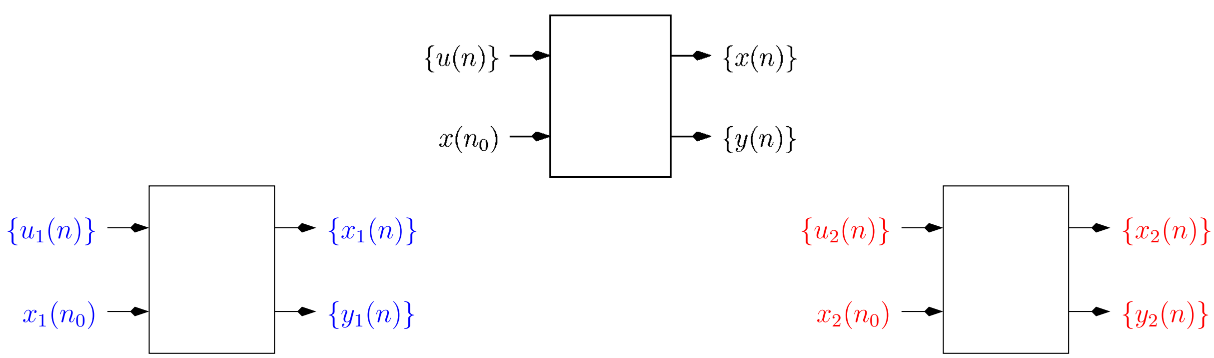
2. We want these to be linear:

   $x(n),\ \ n > 0$: future states
   $y(n),\ \ n > 0$: future outputs

Example (will show to be linear):

$$x(n+1) = Ax(n) + Bu(n)$$
$$y(n) = Cx(n) + Du(n)$$

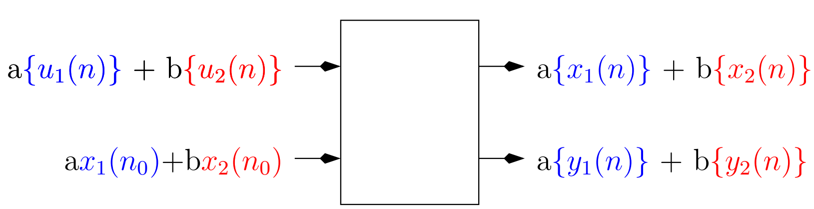
2. Definition of a Linear System



Definition of Linearity of a function $f(x)$:

$$f(\mathrm{a}x_1 + \mathrm{b}x_2) = \mathrm{a}f(x_1) + \mathrm{b}f(x_2)$$

If the following is true, we call the system to be linear



3. Example of a Linear System

Is the following function linear?

$$x(n+1) = A(n)x(n) + B(n)u(n)$$

1. Assume initial state is $x_1(n_0)$, input is $u_1(n)$, calculate $x_1(n_0+1)$:

$$x_1(n_0+1) = A(n_0)x_1(n_0) + B(n_0)u_1(n_0)$$

2. Assume initial state is $x_2(n_0)$, input is $u_2(n)$, calculate $x_2(n_0+1)$:

$$x_2(n_0+1) = A(n_0)x_2(n_0) + B(n_0)u_2(n_0)$$

3. Apply linear combination of initial states

$$x(n_0) = \alpha x_1(n_0) + \beta x_2(n_0)$$

and same combination of inputs

$$u(n) = \alpha u_1(n) + \beta u_2(n)$$

4. Evaluate the next state

$$\begin{aligned} x(n_0+1) &= A(n_0)(\alpha x_1(n_0) + \beta x_2(n_0)) \\ &\quad + B(n_0)(\alpha u_1(n_0) + \beta u_2(n_0)) \end{aligned}$$

Collect the terms

$$\begin{aligned} &= \alpha(A(n_0)x_1(n_0) + B(n_0)u_1(n_0)) \\ &\quad + \beta(A(n_0)x_2(n_0) + B(n_0)u_2(n_0)) \end{aligned}$$

Substitute

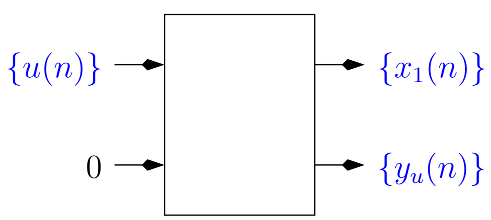
$$= \alpha x_1(n_0+1) + \beta x_2(n_0+1)$$

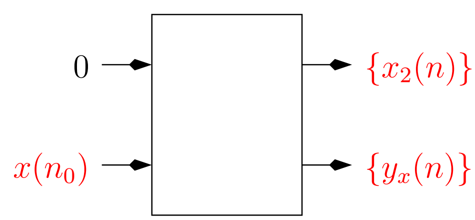
5. Same linear combination as inputs and initial states. Hence linear at n_0.

6. Can apply the same procedure for all future inputs. Hence linear.

Similarly, $y(n) = C(n)x(n) + D(n)u(n)$ is also linear

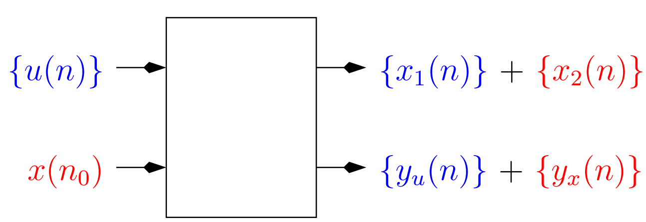
4. $y = y_u + y_x$





y_u: Due to only input $\qquad$ y_x: Due to only initial state

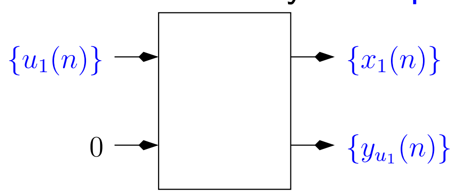
If u is doubled, both x and y double

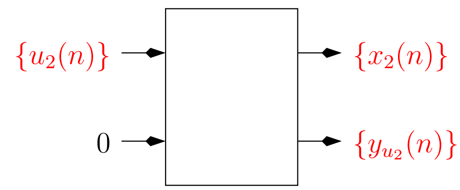


- Output (y) = output due to input (y_u) + output due to initial state (y_x)

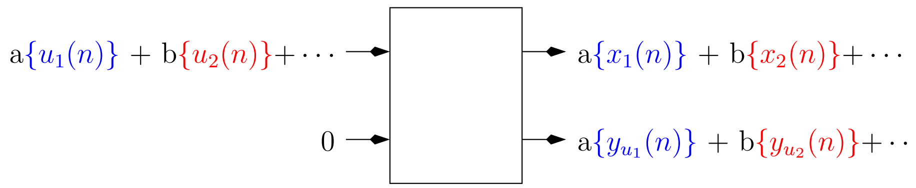
5. How to Achieve Linearity from Input Alone?

Carry out input excitations with zero initial state:

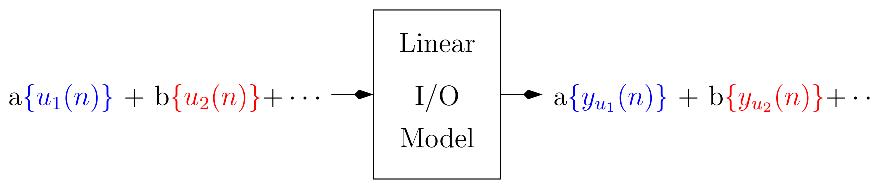




Linear from input only - for example, if we double the input, the output will double - does not happen if the initial value is not zero.



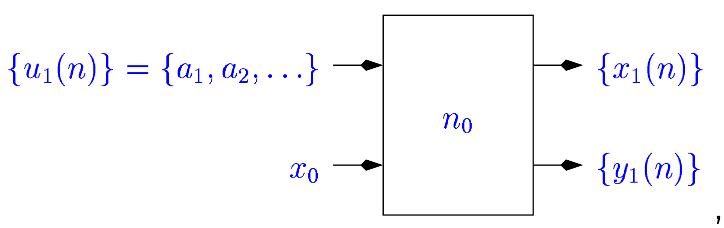
Can drop the initial state information as the system is input/output linear.



Initial State = 0

6. Time Invariant System

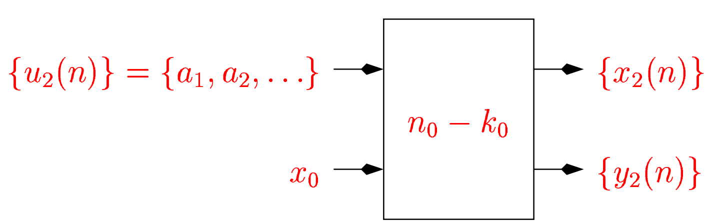
Apply input u to a system at initial state $n = n_0$



,

$n \geq n_0$

Identical input sequence and initial state at $n = n_0 - k_0$:



$n \geq n_0 - k_0$

Given

$$\{u_2(n)\} = \{u_1(n-k)\}$$

Are the following satisfied?

$$\{x_2(n)\} = \{x_1(n-k)\}$$
$$\{y_2(n)\} = \{y_1(n-k)\}$$

If so, system is *time invariant*. Examples:

$$y(n) = nu(n)$$

is not time invariant , while

$$x(n+1) = Ax(n) + Bu(n)$$
$$y(n) = Cx(n) + Du(n)$$

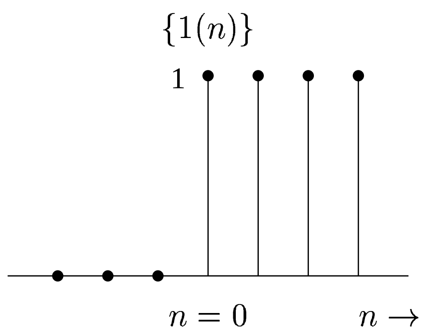
is.

7. Basic Discrete Signals

A unit step sequence is defined as $\{1(n)\} = \{\dots, 1(-2), 1(-1), 1(0), 1(1), \dots\}$:

$$1(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\{1(n)\} = \{\dots, 0, 0, 1, 1, \dots\}$$

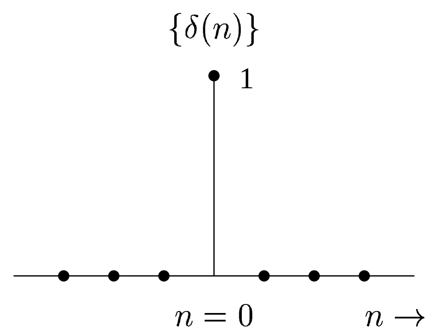


- Takes a value of 1 from the time the argument becomes zero.

Unit impulse sequence or unit sample sequence is defined as $\{\delta(n)\} = \{\dots, \delta(-2), \delta(-1), \delta(0), \delta(1), \dots\}$

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\{\delta(n)\} = \{\dots, 0, 0, 1, 0, 0, \dots\}$$



- Takes a value of 1 at the time the argument becomes zero.

8. Arbitrary Sequence Using δ

$$\{u(n)\} = \{\dots, u(-2), u(-1), u(0), u(1), u(2), \dots\}$$

Split them into individual components:

$$= \dots + \{\dots, 0, 0, u(-2), 0, 0, 0, 0, 0, \dots\} + \{\dots, 0, 0, 0, u(-1), 0, 0, 0, 0, \dots\}$$
$$+ \{\dots, 0, 0, 0, 0, u(0), 0, 0, 0, \dots\} + \{\dots, 0, 0, 0, 0, 0, u(1), 0, 0, \dots\}$$
$$+ \{\dots, 0, 0, 0, 0, 0, 0, u(2), 0, \dots\} + \dots$$

Split each into a scalar and an impulse

$$\begin{aligned} &= \dots + u(-2)\{\delta(n+2)\} + u(-1)\{\delta(n+1)\} \\ &+ u(0)\{\delta(n)\} + u(1)\{\delta(n-1)\} + u(2)\{\delta(n-2)\} + \dots \end{aligned}$$

$$\{u(n)\} = \sum_{k=-\infty}^{\infty} u(k)\{\delta(n-k)\} = \{u(n)\} * \{\delta(n)\}$$

Can show that by comparing the two sides term by term,

$$u(n) = \sum_{k=-\infty}^{\infty} u(k)\delta(n-k) = u(n) * \delta(n)$$

9. Impulse Response Models for LTI Systems

$$\{\delta(n)\} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{I/O Model}\end{array}} \longrightarrow \{g(n)\}$$

Shift the input. As time invariant, output will also be shifted:

$$\{\delta(n-k)\} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{I/O Model}\end{array}} \longrightarrow \{g(n-k)\}$$

$$\{u(n)\} = \sum_{k=-\infty}^{\infty} u(k)\{\delta(n-k)\}$$

$$\{y(n)\} = \sum_{k=-\infty}^{\infty} u(k)\{g(n-k)\}$$

$$\{g(n)\} = \{1, 2, 3\}$$

$$\{u(n)\} = \{4, 5, 6\}$$

Determine output sequence $\{y(n)\}$.
g, u start at $n = 0$.
They are zero for $n < 0$.

$$y(0) = u(0)g(0) = 4$$
$$y(1) = u(0)g(1) + u(1)g(0) = 13$$
$$y(2) = u(0)g(2) + u(1)g(1) + u(2)g(0) = 28$$
$$y(3) = u(1)g(2) + u(2)g(1) = 27$$
$$y(4) = u(2)g(2) = 18$$

All other terms that don't appear above are zero.

- Impulse response has all information about LTI system
- Given impulse response, can determine output due to any arbitrary input

10. Step Response and Relation with Impulse Response

The unit step response of an LTI system at zero initial state $\{s(n)\}$ is the output when $\{u(n)\} = \{1(n)\}$:

$$\{s(n)\} = \sum_{k=-\infty}^{\infty} 1(k)\{g(n-k)\}$$

Apply the meaning of $1(k)$:

$$= \sum_{k=0}^{\infty} \{g(n-k)\}$$

- This shows that the step response is the sum of impulse response.

- We can also get impulse response from step response.

$$\{\delta(n)\} = \{1(n)\} - \{1(n-1)\}$$

Using linearity and time invariance properties,

$$\{h(n)\} = \{s(n)\} - \{s(n-1)\}$$

Can show that

$$\{y(n)\} = [\{u(n)\} - \{u(n-1)\}] * \{s(n)\}$$
$$= \{\triangle u(n)\} * \{s(n)\}$$

11. Causality of LTI Systems

- If output depends only on past inputs, called causal
- If output depends on future inputs, not causal
- For LTI causal systems, $g(n) = 0$ for $n < 0$

12. Output of Causal Systems

- Impulse response $g(n)$ is causal: $g(n) = 0$ for negative n
- Let $u(n) = 0$ for negative n

$$\{y(n)\} = \sum_{k=-\infty}^{\infty} u(k)\{g(n-k)\}$$

Comparing term by term (done in the text)

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} u(k)g(n-k) \\ &= \sum_{k=0}^{\infty} u(k)g(n-k) \quad (u(k) = 0\ \forall k < 0) \\ &= \sum_{k=0}^{n} u(k)g(n-k) \quad (g(k) = 0\ \forall k < 0) - h \text{ is causal} \end{aligned}$$

13. Recursive Solution to Discrete State Space Equation

$$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k) + Du(k),\ \{u(k)\} = \{u(0), u(1), u(2), \ldots\}$$

$$x(1) = Ax(0) + Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = A[Ax(0) + Bu(0)] + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$

$$x(3) = Ax(2) + Bu(2) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)$$

$$x(k) = A^kx(0) + \sum_{i=0}^{k-1} A^{k-(i+1)}Bu(i), \qquad A^o = I$$

$$y(k) = \underbrace{CA^kx(0)}_{\text{state response } y_x} + \underbrace{\sum_{i=0}^{k-1} CA^{k-(i+1)}Bu(i) + Du(k)}_{\text{input response } y_u}$$

In input-output setting, we get

$$y(k) = y_x + y_u = y_x + \sum_{i=0}^{k} u(i)g(k-i) = y_x + \sum_{i=0}^{k-1} u(i)g(k-i) + u(k)g(0)$$

Comparing terms, we get,

$$g(k) = CA^{k-1}B,\ k > 0, \quad g(0) = D$$

Usually, however, D and hence $g(0)$, are zero.

14. Convolution is Commutative

$$u(n) * g(n) = g(n) * u(n)$$

$$u(n) * g(n) = \sum_{k=-\infty}^{\infty} u(k)g(n-k)$$

with $r = n - k$,

$$= \sum_{r=-\infty}^{\infty} u(n-r)g(r)$$

$$= \sum_{r=-\infty}^{\infty} g(r)u(n-r)$$

$$= g(n) * u(n)$$

- Output of an LTI system with input $u(n)$ and unit impulse response $g(n)$ $=$ output with input $g(n)$ and unit impulse response $u(n)$
- This property does not hold even if one of the sequences is either nonlinear or time varying or both

15. Convolution is Associative

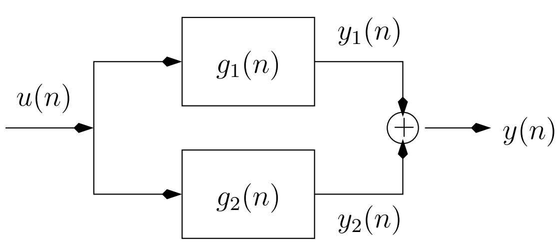
$$
\begin{aligned}
u(n) * (g_1(n) * g_2(n)) &= (u(n) * g_1(n)) * g_2(n) \\
L.H.S. &= u(n) * \sum_{k=-\infty}^{\infty} g_1(n-k) g_2(k) \\
&= \sum_{r=-\infty}^{\infty} u(r) \sum_{k=-\infty}^{\infty} g_1(n-k-r) g_2(k) \\
R.H.S. &= \left(\sum_{r=-\infty}^{\infty} u(r) g_1(n-r) \right) * g_2(n) \\
&= \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} u(r) g_1(n-r-k) g_2(k) = L.H.S.
\end{aligned}
$$

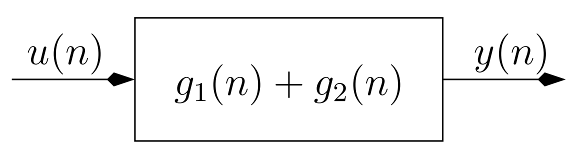
As a result of this relation, we can write

$$
\begin{aligned}
u(n) * (g_1(n) * g_2(n)) &= (u(n) * g_1(n)) * g_2(n) \\
&= u(n) * g_1(n) * g_2(n)
\end{aligned}
$$

16. Convolution Distributes over Addition

$$
\begin{aligned}
u(n) * (g_1(n) + g_2(n)) &= u(n) * g_1(n) + u(n) * g_2(n) \\
L.H.S. &= \sum_{r=-\infty}^{\infty} u(n-r)(g_1(r) + g_2(r)) \\
&= \sum_{r=-\infty}^{\infty} u(n-r) g_1(r) + \sum_{r=-\infty}^{\infty} u(n-r) g_2(r) \\
&= u(n) * g_1(n) + u(n) * g_2(n)
\end{aligned}
$$





- Similar relations hold true for causal systems as well.