1. Example

Recall previous example. Eigenvalues of A are 1, 3:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{aligned} A - bK &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -K_1 & 3 - K_2 \end{bmatrix} \end{aligned}$$

If $K_1 = 0.5$, $K_2 = 3.5$,

$$A - bK = \begin{bmatrix} 1 & 2 \\ -0.5 & -0.5 \end{bmatrix}$$

- Eigenvalues are $0.25 \pm 0.6614j$. Stable.

Now use $K = e_n^T \mathcal{C}^{-1} \alpha_c(A)$

2. Example

$$\begin{aligned} \alpha_c &= (z - (0.25 + j0.6614))(z - (0.25 - j0.6614)) \\ &= z^2 - 0.5z + 0.5 \end{aligned}$$

$$\begin{aligned} \alpha_c(A) &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - 0.5 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 0.5 & 1 \\ 0 & 1.5 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix} \end{aligned}$$

$$\mathcal{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}, \mathcal{C}^{-1} = -\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}$$

$$\begin{aligned} K &= e_n^T \mathcal{C}^{-1} \alpha_c(A) = -\frac{1}{2} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 7 \end{bmatrix} = \begin{bmatrix} 0.5 & 3.5 \end{bmatrix} \end{aligned}$$

Agree with the previous values

3. Controllability

- Can place poles at any place, provided $\mathcal{C} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$ is nonsingular
- Equivalently, the system can be made arbitrarily fast, by placing poles at suitable locations
- If $\mathcal{C}$ is nonsingular, the system is said to be Controllable
- Example of a controllable system:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}.$$

$\mathcal{C}$ is nonsingular and hence system is controllable

4. Controllability

- Example of an uncontrollable system:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

- $\mathcal{C}$ is singular and hence the system is uncontrollable.
- Physical interpretation:

$$x(m+1) = Ax(m) + bu(m)$$
$$\begin{bmatrix} x_1(m+1) \\ x_2(m+1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(m) \\ x_2(m) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

- The control action does not affect x_2 at all!
- Unfortunately, x_2 is also unstable.

5. What Happens to Controller if States are Not Measured?

- State feedback that we used earlier:

$$u = -Kx + v.$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u = -Kx = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Suppose that all states are not measured.
- Suppose only one measurement is made.

6. What Happens to Controller if States are Not Measured?

- If only x_1 is measured

$$y = x_1 = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = cx,$$
$$c = [1 \;\; 0]$$

- If only x_2 is measured

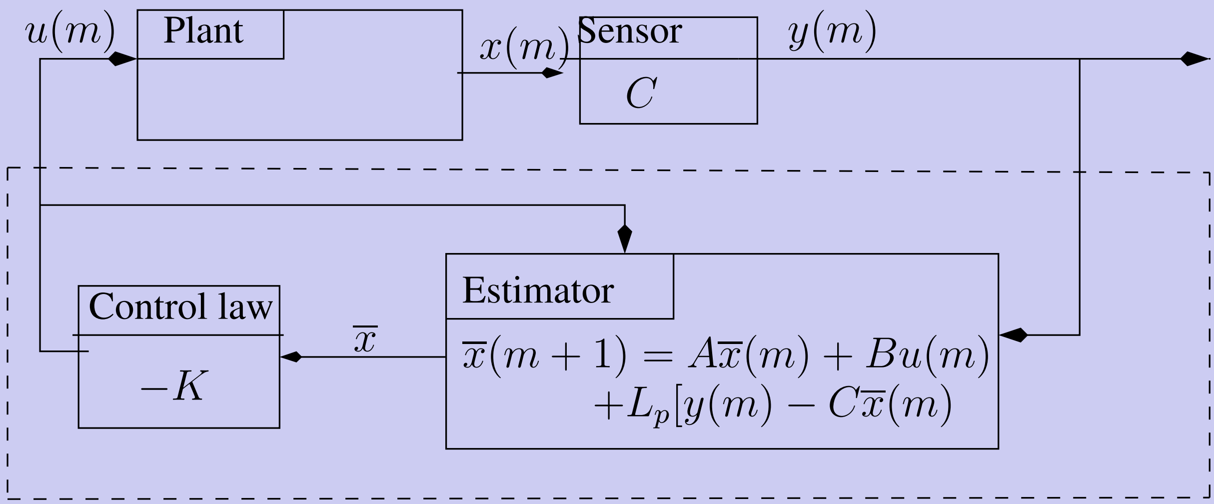
$$y = x_2 = [0 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = cx,$$
$$c = [0 \;\; 1]$$

- If a sum is measured

$$y = x_1 + x_2 = [1 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = cx,$$
$$c = [1 \;\; 1]$$

- In general, fewer than n states are measured.

7. Use Estimator to Determine States



The structure of the estimator will be explained shortly.

8. States are Not Available - Estimate Them

- If in $x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1)$, not all states are not measured, it is equivalent to not knowing initial state x_0.
- Because, if the initial state is known,

  $$x(1) = Ax(0) + bu(0)$$

  $x(1)$ can be calculated.

  $$x(2) = Ax(1) + bu(1)$$

  $x(2)$ can be calculated. All future states can be calculated recursively.
- All states are not measured = initial state unknown.

9. An Unsuccessful Approach to Estimator Design

- Plant:

$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1)$$
$$y(m) = cx(m)$$

- Will the following work as the estimator?

$$\overline{x}(m+1) = A\overline{x}(m) + bu(m) + \overline{x}_0\delta(m+1)$$

where, $\overline{x}$ is a calculated estimate of x.

- Subtract and let $e = \overline{x} - x$:

$$e(m+1) = Ae(m) + e_0\delta(m+1)$$

How does this error dynamics behave?

10. Open Loop Analysis: Controller yet to be Designed

Recall $\overline{x}$, x, $e = \overline{x} - x$:

$$\overline{x}(m+1) = A\overline{x}(m) + bu(m) + \overline{x}_0\delta(m+1)$$
$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1)$$
$$e(m+1) = Ae(m) + e_0\delta(m+1)$$

- Error dynamics governed by eigenvalues of A only
  - If inside unit circle, e will go to zero
  - e is zero, because it is of the form $e = \overline{x} - x = 0 - 0$
  - If outside, $e(m+1) = \infty - \infty = \infty$, as $k \to \infty$.
- In fact, we would like $e(m)$ to behave in a nice way: fast rise time, small overshoot, etc.
- Cannot use the above as the estimator.

11. A Potential Estimator?

- Plant:

$$
\begin{aligned}
x(m+1) &= Ax(m) + bu(m) + x_0\delta(m+1) \\
y(m) &= cx(m)
\end{aligned}
$$

- Will the following work as the estimator?

$$
\begin{aligned}
\overline{x}(m+1) = A\overline{x}(m) &+ bu(m) + \overline{x}_0\delta(m+1) \\
&+ L_p[y(m) - c\overline{x}(m)]
\end{aligned}
$$

  where, L_p is to be calculated.

- Subtract and let $e = \overline{x} - x$:

$$
e(m+1) = Ae(m) + e_0\delta(m+1) + L_p[y(m) - c\overline{x}(m)]
$$

  Can we make the error dynamics behave the way we want?

12. A Potential Estimator?

Recall equations for $\overline{x}$ and x, $e = \overline{x} - x$, $y = cx$

$$
\begin{aligned}
\overline{x}(m+1) &= A\overline{x}(m) + bu(m) + \overline{x}_0\delta(m+1) + L_p[y(m) - c\overline{x}(m)] \\
x(m+1) &= Ax(m) + bu(m) + x_0\delta(m+1) \\
e(m+1) &= Ae(m) + e_0\delta(m+1) + L_p[y(m) - c\overline{x}(m)] \\
&= Ae(m) + e_0\delta(m+1) + L_p[cx(m) - c\overline{x}(m)] \\
&= Ae(m) + e_0\delta(m+1) + L_pc[x(m) - \overline{x}(m)] \\
&= Ae(m) + e_0\delta(m+1) - L_pce(m) \\
&= (A - L_pc)e(m) + e_0\delta(m+1)
\end{aligned}
$$

- A is unstable
- Can I choose L_p so as to make $(A - L_pc)$ stable and well behaved?

13. Regulator - Estimator Duality

- Pole placement problem:

$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1)$$

We chose K so that $(A - bK)$ is well behaved, that is,

$|zI - (A - bK)| = \alpha_c(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n =$

- Solution: $K = e_n^T \mathcal{C}^{-1}\alpha_c(A)$.
- Estimator problem:

$$e(m+1) = (A - L_p c)e(m) + e_0\delta(m+1)$$

A not well behaved. Can we choose L_p so that $(A - L_p c)$ is well behaved? i.e.

$$|zI - (A - L_p c)| = \alpha_e(z)$$

Want eigenvalues of $A - L_p c$ to be at desirable locations

14. Regulator - Estimator Duality

- Pole placement problem:

$$x(m+1) = Ax(m) + bu(m) + x_0\delta(m+1)$$

We chose K so that $(A - bK)$ is well behaved.

- Estimator problem:

$$e(m+1) = (A - L_p c)e(m) + e_0\delta(m+1)$$

Can we choose L_p so that $(A - L_p c)$ is well behaved?

- K and L_p are at different locations. Can't use directly.
- But, eigenvalues of $(A - L_p c)$ = eigenvalues of $(A - L_p c)^T$ = eigenvalues of $A^T - c^T L_p^T$.
- Now the unknowns are in the same location!

15. Ackermann's Formula for Estimator Design

To place eigenvalues of $A - bK$ at required locations,

$$\begin{aligned} |zI - (A - bK)| &= \alpha_c(z) \\ &= z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n \\ &= (z - \beta_1)(z - \beta_2) \cdots (z - \beta_n) \\ K &= e_n^T \mathcal{C}^{-1} \alpha_c(A) \\ \mathcal{C} &= \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}. \end{aligned}$$

To place eigenvalues of $A^T - c^T L_p^T$ at desirable locations,

$$|zI - (A^T - c^T L_p^T)| = \alpha_e(z),$$

Use the formula,

$$L_p^T = e_n^T [\mathcal{O}^T]^{-1} \alpha_e(A^T)$$

16. Ackermann's Formula for Estimator Design

Estimator design:

$$L_p^T = e_n^T [\mathcal{O}^T]^{-1} \alpha_e(A^T)$$

Assume the following matrix to be nonsingular:

$$\mathcal{O}^T = \begin{bmatrix} c^T & A^T c^T & \cdots (A^T)^{n-1} c^T \end{bmatrix}$$

Taking transpose of first equation,

$$\begin{aligned} L_p &= \alpha_e(A) \mathcal{O}^{-1} e_n \\ &= \alpha_e(A) \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}^{-1} e_n \end{aligned}$$