1. Auto Covariance Function - Example

Find the ACF of \( \{u(n)\} = \{1, 2\} \). Mean = \( m_u = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u(n) \)

\[
\begin{align*}
m_u &= \frac{1}{2} \sum_{k=0}^{1} u(k) = \frac{1}{2} (u(0) + u(1)) = 1.5
\end{align*}
\]

ACF = \( r_{uu}(l) = \lim_{N \to \infty} \frac{1}{2N} \sum_{k=-N}^{N} (u(k) - m_u)(u(k - l) - m_u) \)

\[
\begin{align*}
r_{uu}(0) &= \sum_{k=0}^{1} (u(k) - 1.5)^2 = (0.5)^2 + 0.5^2 = 0.5 \\
r_{uu}(1) &= \sum_{k=0}^{1} (u(k) - 1.5)(u(k - 1) - 1.5) \\
&= (u(1) - 1.5)(u(0) - 1.5) = 0.5 \times (-0.5) = -0.25 \\
r_{uu}(-1) &= \sum_{k=0}^{1} (u(k) - 1.5)(u(k + 1) - 1.5) \\
&= \frac{1}{2} (u(0) - 1.5)(u(1) - 1.5) = (-0.5) \times 0.5 = -0.25
\end{align*}
\]

2. Cross Covariance Function - Example

Estimate of CCF:

\[
\begin{align*}
r_{uy}(l) &= \lim_{N \to \infty} \frac{1}{2N} \sum_{k=-N}^{N} (u(k) - m_u)(y(k - l) - m_y)
\end{align*}
\]

To calculate \( r_{uy}(l), l > 0 \), shift \( y \) by \( l \) points to right multiply, add. Check:

\[
r_{uy}(l) = r_{yu}(-l)
\]

Causality: Current output cannot be correlated with a future input.

\[
r_{uy}(l) = r_{yu}(-l) = 0, \ \forall l < 0
\]
3. White Noise

- The discrete-time white-noise sequence \( \{ \xi(k) \} \) is a set of independent, identically distributed (i.i.d.) values belonging to a stationary stochastic process.
- The mean of white noise is zero.
- The ACF of a white-noise sequence is given by:
  \[
  \gamma_{\xi \xi}(k) = \sigma_{\xi}^2 \delta(k) = \begin{cases} 
    \sigma_{\xi}^2 & k = 0 \\
    0 & \text{otherwise}
  \end{cases}
  \]
- The Z-transform of ACF of white noise is \( \sigma_{\xi}^2 \).
- White noise has infinite energy \( \Rightarrow \) its Fourier Transform does not exist.
- But the Fourier Transform of ACF (power spectrum) of white-noise is constant and given by,
  \[
  \Phi_{\xi \xi}(\omega) = \sigma_{\xi}^2, \quad \forall \omega
  \]

4. Use of ACF: An Example

Determine \( a \) with white noise \( u \):
\[
y(n) - ay(n-1) = \xi(n), \quad (1)
\]
Multiply Eq. 1 by \( \xi(n-k) \) and sum:
\[
\gamma_{y\xi}(k) - a \gamma_{y\xi}(k-1) = \gamma_{\xi \xi}(k). \quad (2)
\]
Because the system is causal,
\[
\gamma_{y\xi}(n) = 0, \quad \forall n < 0.
\]
By evaluating Eq. 2 for \( k = 0 \) and 1,
\[
\begin{align*}
\gamma_{y\xi}(0) &= \gamma_{\xi \xi}(0) = \sigma_{\xi}^2, \\
\gamma_{y\xi}(1) &= a \sigma_{\xi}^2.
\end{align*}
\]
Multiply Eq. 1 by \( y(n-k) \) and sum:
\[
\gamma_{yy}(k) - a \gamma_{yy}(k-1) = \gamma_{y\xi}(-k)
\]
Evaluate for \( k = 0, 1 \):
\[
\begin{align*}
\gamma_{yy}(0) - a \gamma_{yy}(1) &= \sigma_{\xi}^2 \\
\gamma_{yy}(1) - a \gamma_{yy}(0) &= 0
\end{align*}
\]
Solving,
\[
a = \frac{\gamma_{yy}(1)}{\gamma_{yy}(0)} = \frac{\sigma_{\xi}^2}{1 - a^2}
\]
From top equation,
\[
\gamma_{yy}(k) = a \gamma_{yy}(k-1) = a^2 \gamma_{yy}(k-2) = \cdots = \gamma_{yy}(0) a^k
\]
5. ACF of Noisy Periodic Signal is Periodic

\[
\sin 0.1n + 0.1\xi(n):
\]

\[
\sin 0.1n + \xi(n):
\]

6. ARMA, AR Processes

ARMA: Auto Regressive Moving Average. Required to model noise process.

\[
y(n) + a_1y(n-1) + \cdots + a_py(n-p) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q)
\]

If \( q = 0 \), we obtain an AR(\( p \)) process:

\[
y(n) + a_1y(n-1) + \cdots + a_py(n-p) = \xi(n) = \left( 1 + \sum_{k=1}^{p} a_kz^{-k} \right)y(n)
\]

or, equivalently,

\[
y(n) = \frac{1}{1 + \sum_{k=1}^{p} a_kz^{-k}}\xi(n) = \frac{1}{A(z)}\xi(n)
\]

- A random sequence whose value \( y(n) \) can be represented as a weighted finite aggregate of the \( p \) previous values plus a white-noise sequence \( \xi(n) \) is said to be an AR process of order \( p \).
7. ARMA, MA Processes

Recall ARMA process:

\[ y(n) + a_1y(n-1) + \cdots + a_py(n-p) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q) \]

When \( p = 0 \), arrive at MA(\( q \)) process:

\[ y(n) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q) \]

\[ = \left( 1 + \sum_{k=1}^{q} c_kz^{-k} \right) \xi(n) = C(z)\xi(n) \]

- A random sequence whose value \( y(n) \) can be represented as a finite combination of the past white-noise sequence \( e \) plus a random error \( \xi(n) \) is said to be an MA process of order \( q \).

ARMA contains both AR and MA components. Using the above procedure,

\[ y(n) = \frac{C(z)}{A(z)}\xi(n) = \frac{1 + \sum_{n=1}^{q} c_nz^{-n}}{1 + \sum_{n=1}^{p} a_nz^{-n}}\xi(n) \]

8. Procedure to Distinguish MA(1) Processes

Develop a method to determine the order of the MA(1) process:

\[ y(k) = \xi(k) + c_1\xi(k-1). \]

We will begin with the calculation of ACF at zero lag:

\[ \gamma_{yy}(0) = \mathcal{E}(y(k), y(k)) = \mathcal{E}[(\xi(k) + c_1\xi(k-1))(\xi(k) + c_1\xi(k-1))] \]

Because \( \xi(k) \) is white, expectation of cross products are zero. We obtain

\[ \gamma_{yy}(1) = (1 + c_1^2)\sigma_\xi^2 \]

Next we determine ACF at lag 1:

\[ \gamma_{yy}(1) = \mathcal{E}(y(k), y(k-1)) = \mathcal{E}[(\xi(k) + c_1\xi(k-1))(\xi(k-1) + c_1\xi(k-2))] \]

Once again invoking the fact that \( e \) is white and cancelling the cross terms, we obtain

\[ \gamma_{yy}(1) = \mathcal{E}(c_1\xi^2(k-1)) = c_1\sigma_\xi^2 \]

For all other lags, ACF is zero. That is,

\[ \gamma_{yy}(l) = 0, \quad l > 1 \]
9. Procedure to Distinguish MA($q$) Processes

Start with a general MA($q$) process:
\[ y(n) = \xi(n) + c_1\xi(n-1) + \cdots + c_q\xi(n-q) \]

Multiplying by $y(n)$ and taking expectation
\[ \gamma_{yy}(0) = \gamma_{y\xi}(0) + c_1\gamma_{y\xi}(1) + \cdots + c_q\gamma_{y\xi}(q) \]

Multiplying by $y(n-1)$ and taking expectation,
\[ \gamma_{yy}(1) = c_1\gamma_{y\xi}(0) + c_2\gamma_{y\xi}(1) + \cdots + c_q\gamma_{y\xi}(q-1) \]

noting
\[ \mathcal{E}[y(n-1)\xi(n)] = 0 \]

from causality principle: for causal systems, the output cannot depend on future input $\xi(n)$.
Continuing the above process and stacking the resulting equations, we arrive at
\[
\begin{bmatrix}
\gamma_{yy}(0) \\
\gamma_{yy}(1) \\
\vdots \\
\gamma_{yy}(q)
\end{bmatrix} =
\begin{bmatrix}
1 & c_1 & \cdots & c_{q-1} & c_q \\
c_1 & c_2 & \cdots & c_q & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_q & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\gamma_{y\xi}(0) \\
\gamma_{y\xi}(1) \\
\vdots \\
\gamma_{y\xi}(q)
\end{bmatrix}
\]

All the terms below the diagonal are zero. It is clear that
\[ \gamma_{yy}(k) = 0, \ \forall k > q \]

Thus we obtain the rule that for MA($q$) process, a plot of $\{\gamma_{yy}(k)\}$ vs. $k$ becomes zero for all $k > q$.

10. Example: MA(2) Process

Calculate $\{\gamma_{yy}(k)\}$ for
\[ y(n) = \xi(n) + \xi(n-1) - 0.5\xi(n-2) \quad (3) \]

Multiplying by $y(n-k)$, $k \geq 0$
\[ \gamma_{yy}(0) = \gamma_{y\xi}(0) + \gamma_{y\xi}(1) - 0.5\gamma_{y\xi}(2) \]
\[ \gamma_{yy}(1) = \gamma_{y\xi}(0) - 0.5\gamma_{y\xi}(1) \]
\[ \gamma_{yy}(2) = -0.5\gamma_{y\xi}(0) \]
\[ \gamma_{yy}(k) = 0, \ k \geq 0 \]

Multiply (3) by $\xi(n)$, $\xi(n-1)$ and $\xi(n-2)$ and take expectation:
\[ \gamma_{y\xi}(0) = \gamma_{y\xi}(0) = \sigma^2_\xi \]

because $\{\xi(n)\}$ is white.

\[ \gamma_{y\xi}(1) = \gamma_{y\xi}(1) = \sigma^2_\xi \]
\[ \gamma_{y\xi}(2) = -0.5\gamma_{y\xi}(0) = -0.5\sigma^2_\xi \]

Substituting these in Eq. (4),
\[ \gamma_{yy}(0) = (1 + 1 + 0.25)\sigma^2_\xi = 2.25\sigma^2_\xi \]
\[ \gamma_{yy}(1) = (1 - 0.5)\sigma^2_\xi = 0.5\sigma^2_\xi \]
\[ \gamma_{yy}(2) = -0.5\sigma^2_\xi \]
\[ \gamma_{yy}(k) = 0, \ k \geq 3, \]

as expected. Known as theoretical prediction approach.
11. Matlab Code to Calculate ACF

```matlab
1. % Define the model
2. m = idpoly(1,[],[1,1,−0.5]);
3. % Generate noise and the response
4. e = 0.1*randn(100000,1);
5. y = sim(m,e); z = [y e];
6. % Plot noise and plant output
7. subplot(2,1,1), plot(y(1:500)), title('Plant output, noise input vs. time', 'FontSize',14)
8. ylabel('Plant output y', 'FontSize',14)
9. subplot(2,1,2), plot(e(1:500)), ylabel('Noise input e', 'FontSize',14)
10. xlabel('Sampling instant, k', 'FontSize',14)
11. % Calculate covariance and plot it
12. ryy = xcov(y,'coeff');
13. figure, plotacf(y,1,11,1);
```

12. Input-Output Plots and ACF

![Plant output and noise input vs. time](image1)

![Noise input](image2)

![ACF Lag](image3)